

On Kostant sections and topological nilpotence

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ABSTRACT

Let G denote a connected, quasi-split reductive group over a field F that is complete with respect to a discrete valuation and that has a perfect residue field. Under mild hypotheses, we produce a subset of the Lie algebra $\mathfrak{g}(F)$ that picks out a $G(F)$ -conjugacy class in every stable, regular, topologically nilpotent conjugacy class in $\mathfrak{g}(F)$. This generalizes an earlier result obtained by DeBacker and one of the authors under stronger hypotheses. We then show that if F is p -adic, then the characteristic function of this set behaves well with respect to endoscopic transfer.

1. Introduction

Let G be a (quasi-split) connected reductive group over a field F , with Lie algebra \mathfrak{g} . In [20], assuming F to be an algebraically closed field of characteristic zero, Kostant gave a simple, elegant and remarkably useful recipe to construct sections (now called *Kostant sections*) to the geometric invariant theory (GIT) quotient $\mathfrak{g} \rightarrow \mathfrak{g}/G$, taking values in the set of regular elements of \mathfrak{g} . When G is a general linear group, the sections thus constructed include as a special case the companion matrix found in elementary linear algebra. In this paper, taking F to be a p -adic field (or more generally a complete discrete valuation field), we make the case that constructing such sections integrally and studying them help us better understand a certain subset of $\mathfrak{g}(F)$ that has shown up in certain problems related to harmonic analysis on p -adic groups. Based on these results we exhibit new examples of pairs of functions that match each other in the sense of endoscopic transfer for Lie algebras.

Let F be a complete discrete valuation field, and let $p \geq 0$ denote the characteristic of the residue field κ , which we assume to be perfect. Let G be a quasi-split, connected, reductive group over F , and $Y \in \mathfrak{g}(F) = \text{Lie } G(F)$ a regular nilpotent element. The first result that we prove in this paper is the following statement that sharpens and extends the main result of [4]. Under mild hypotheses (see later in this introduction), we present a neighborhood of Y — let us call it $Y + \mathfrak{g}_{x,0+}$ in view of notation that will be established later — with the following two properties:

- (a) if $X \in \mathfrak{g}(F)$, then the $\text{Ad } G(\overline{F})$ -orbit of X intersects $Y + \mathfrak{g}_{x,0+}$ if and only if X is regular and topologically nilpotent; and
- (b) if $X \in \mathfrak{g}(F)$ is regular and topologically nilpotent, then the intersection $\text{Ad } G(\overline{F})(X) \cap (Y + \mathfrak{g}_{x,0+})$ is a single $\text{Ad } G_{x,0+}$ -orbit, for a fixed (independent of X) bounded open subgroup $G_{x,0+} \subset G(F)$.

As the notation suggests, in (a) and (b) above, x stands for a certain point in the Bruhat–Tits building of G , and $\mathfrak{g}_{x,0+}$ (respectively, $G_{x,0+}$) is the associated Moy–Prasad lattice (respectively, subgroup).

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The characteristic function $\phi = \mathbb{1}_{Y + \mathfrak{g}_{x,0+}}$ of the set $Y + \mathfrak{g}_{x,0+}$, like its several variants mentioned below, has been no stranger to harmonic analysis, partly because of the role played by nilpotent elements in representation theory, and partly because orbital integrals, whose evaluation on ϕ is facilitated by (b) above, are important to harmonic analysis. For instance, when $G = \mathrm{GL}_n$ over a p -adic field F , J. Repka used the function ϕ (or rather, its composite with $g \mapsto g - 1$) in order to compute Shalika germs associated to the regular unipotent conjugacy class of $G(F)$ (see [27]). More general variants of this set have shown up in many important works on the subject, especially in the context of character theory and nilpotent orbits — for instance the set $n_\alpha + L_j$ in [17, the proof of Lemma 6], its generalization $Y + \varpi^n L'^\perp$ in [24, the proof of Proposition I.11], and the sets $X + \mathfrak{g}_{F^*}^+$ in [12].

As alluded to earlier, DeBacker and the first-named author proved (see [4, Proposition 1]) the aforementioned result on the properties (a) and (b) of the set $Y + \mathfrak{g}_{x,0+}$ under more restrictive hypotheses, and only for regular semisimple elements X . The hypotheses of [4] require that certain reductive groups over finite fields admit suitably well-behaved \mathfrak{sl}_2 -triples. For example, if $G = \mathrm{Sp}_{2n}$, then the use of [11, Hypothesis 4.2.3] requires p to be at least $4n + 1$ if nonzero. Our result, by contrast, is always valid for $p > 5$ if G is semisimple, tamely ramified and has no factor of type A_n . Moreover, if $G = \mathrm{GL}_n$, then we impose no restriction on p . We hope that our new presentation, in addition to weakening hypotheses, makes certain aspects of the role of the Kostant section more explicit.

The second result says that, if F is a finite extension of \mathbb{Q}_p and p satisfies a few further hypotheses, then the function ϕ behaves well with respect to endoscopic transfer. In other words, suppose that the conditions of Hypothesis 36 are satisfied by G as well as by a group H that is endoscopic for G , and that $\phi_H = \mathbb{1}_{Y_H + \mathfrak{h}_{x_H,0+}} \in C_c^\infty(\mathfrak{h}(F))$ is the function obtained by applying the construction of ϕ to H in place of G . The statement then is that, up to an explicitly computable nonzero scalar, ϕ and ϕ_H have matching orbital integrals. This fact can be used to cook up more pairs of functions with matching orbital integrals. The interest in such results comes from the fact that the theory of endoscopy uses orbital integrals to relate harmonic analysis on $H(F)$ with that on $G(F)$, but the supply of explicit pairs of functions with matching orbital integrals in the literature is somewhat limited (for an example of some deep work on this question, see the paper [19] by Kazhdan and Varshavsky).

Now let us remark on some considerations that motivated our proof of (a) and (b) above. The proof of [4, Proposition 1] uses a hypothesis that Y can be completed to an \mathfrak{sl}_2 -triple containing another nilpotent element Y' such that the following equation holds (as also its analogues over finite extensions of F)

$$Y + \mathfrak{g}_{x,0+} = \mathrm{Ad} G_{x,0+} (Y + C_{\mathfrak{g}_{x,0+}}(Y')) ,$$

where $C_{\mathfrak{g}_{x,0+}}(Y')$ is the centralizer of Y' in $\mathfrak{g}_{x,0+}$. Note that $Y + C_{\mathfrak{g}_{x,0+}}(Y')$ is part of the Kostant section $Y + C_{\mathfrak{g}(F)}(Y')$ (cf. [20]) attached to Y and Y' . Thus, the assertion of [4] is that $Y + C_{\mathfrak{g}_{x,0+}}(Y')$ is precisely the set of topologically nilpotent elements in the Kostant section $Y + C_{\mathfrak{g}(F)}(Y')$.

Now suppose G is unramified. Then x is hyperspecial and gives a realization of G as a reductive group over the ring \mathfrak{O} of integers of F . In this case (for p satisfying the hypotheses of [11]), $Y + C_{\mathfrak{g}_{x,0}}(Y')$ is a Kostant section for G over \mathfrak{O} , from which the above claim about $Y + C_{\mathfrak{g}_{x,0+}}(Y')$ follows easily. Moreover, as alluded to above, one may make this argument work for p satisfying much weaker hypotheses, at least provided we replace $C_{\mathfrak{g}_{x,0}}(Y')$ by a different \mathfrak{O} -submodule of $\mathfrak{g}_{x,0}$ (compare with [20, Remark 19]).

However, since the main result of [4] is valid even when G is not unramified, one might wish for a proof that still explicitly incorporates the above idea and yet works for ramified groups, at least under mild hypotheses. This is what we do here in § 2.

A few remarks on the hypotheses

Several results that we prove in §2 and §3 of this paper do not hold for all reductive groups G and all residue field characteristics p . In order to nevertheless state our results in large generality, we will need to use the following conditions (see Definitions 15, 17, and 31, and Hypothesis 36):

- that of p being \mathfrak{g} -good — namely, a ‘good prime’ for G in the well-known sense of [32];
- that of p being \mathfrak{n}^- -good — a certain condition weaker than \mathfrak{g} -good;
- that of p being \mathfrak{g} - F -good — a variation on ‘ \mathfrak{g} -good’ adapted to the graded Lie algebra associated to a suitable Moy–Prasad filtration on $\mathfrak{g}(F)$;
- that of G being not too wild — roughly speaking, tameness conditions on the absolutely simple groups that G is built from, and on the ‘interaction’ between T and $T_{\text{der}} := T \cap G_{\text{der}}$, where T is a maximal torus of G and G_{der} is the derived group of G ;
- that of G satisfying condition (T) — that is, a maximal torus of G becomes an induced torus after passing to a tamely ramified extension of F (this ensures the existence of mock exponential maps);
- that of two groups G and H satisfying Hypothesis 36 — a set of conditions designed to ensure a few extra conveniences such as the existence of a Kazhdan–Varshavsky quasi-logarithm.

Note that things simplify considerably when G is split over a tamely ramified extension of F . In this case, G is automatically not too wild and satisfies condition (T). Further, in this case, p is \mathfrak{g} - F -good if and only if it is \mathfrak{g} -good.

Suppose G is defined and split over \mathfrak{O} . Our construction of a candidate for being named a ‘Kostant’ section to the adjoint quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/G$ over \mathfrak{O} already requires p to be \mathfrak{n}^- -good, and this condition turns out to suffice to guarantee the existence of such sections. The assertion (a) stated early on in this introduction requires G to be not too wild and p to be \mathfrak{n}^- -good, while assertion (b) requires that p is in fact \mathfrak{g} - F -good and that G also satisfy condition (T). As stated earlier, Hypothesis 36 shows up and is required only in §3, for our results involving endoscopy.

Structure of the paper

After setting up basic notation (§2.1) and recalling a definition of topological nilpotence (§2.2), we construct in §2.3 an integral version of the Kostant section when G is split, under hypotheses somewhat milder than those in [28] (see Hypothesis 7). In fact, our Kostant section is built using a certain \mathfrak{O} -module, and we see that whenever this module exists, we do indeed obtain a Kostant section (Proposition 10). The main objects that concern us (G , x , Y , etc.) are set up in §2.5, using constructions recalled in §2.4. In §2.6, we define what it means for the residual characteristic p of F to be ‘ \mathfrak{g} -good’, ‘ \mathfrak{g} - F -good’, or ‘ \mathfrak{n}^- -good’ for G . These conditions assure that $\text{ad } Y$ has good regularity properties over the residue field κ of F .

In §2.7, we first consider the case where G is split over a tamely ramified extension E of F , and use an integral Kostant section $Y + L_{\mathfrak{O}_E}$ for the base change of G to E , together with Lemma 22, to get a Kostant section $Y + L_F$ for G (over F), whose subset of topologically nilpotent elements is precisely a set of the form $Y + (L_F \cap \mathfrak{g}_{x,0+})$ in a sense alluded to above. In the same section, under the assumption that p is ‘ \mathfrak{g} - F -good’, we extend this last assertion to G satisfying the weaker condition of being ‘not too wild’. In fact, this last condition is necessary in order to have an F -subspace L_F that is compatible with the \mathfrak{O}_E -module $L_{\mathfrak{O}_E}$. The Kostant section supplied by §2.7 is related to $Y + \mathfrak{g}_{x,0+}$ in §2.8 under the condition that p is \mathfrak{g} - F -good for G , when G satisfies condition (T). Our generalization of the main result of [4] is then deduced in §2.9.

The assertion about endoscopic transfer is proved in §3.7. Since the notion of topological nilpotence behaves well with respect to the matching of conjugacy classes in the theory of endoscopy, there are two main issues to take care of: the transfer factors and normalized orbital integrals. It turns out that [22, Theorem 5.1], recalled along with a review of endoscopic transfer for Lie algebras in §3.4, immediately tells us how to handle the former. While the property (b) of the set $Y + \mathfrak{g}_{x,0+}$ discussed near the beginning of this introduction makes it believable that the orbital integrals can be easily handled, one needs to do a bit more work since different elements in this set generally have nonisomorphic centralizers. However, it turns out that the $\mathrm{Ad} G(F)$ -orbits of elements of $Y + \mathfrak{g}_{x,0+}$ have measures arising from Kirillov’s symplectic forms on them, which are easily evaluated on the intersection of $Y + \mathfrak{g}_{x,0+}$ with these orbits, thanks to Y and $\mathfrak{g}_{x,0+}$ being conveniently adapted to these forms (see §3.3, particularly Lemma 44). Executing this requires passage to the Lie algebra, to which end we impose a stronger hypothesis on p (Hypothesis 36) so as to use a Kazhdan–Varshavsky quasi-logarithm; see §3.5. Further, in §3.2, specifically Lemma 42, we relate the measures arising from Kirillov’s symplectic form construction with a different set of choices for these measures that is fixed in §3.1, the latter being better suited for studying endoscopic transfer. This lets us finish the computation of the orbital integrals in §3.6. In fact, it is our comparison between measures in Lemma 42 that accounts for the ‘ Δ_{IV} ’ transfer factors, or equivalently, the normalization of the orbital integrals. Finally, §3.8 discusses how standard techniques allow us to cook up more pairs of matching functions starting from $(\phi, \phi_{\mathrm{H}})$ as above.

2. Topologically nilpotent elements in a Kostant section

2.1. Basic notation

Let F be a complete, discretely valued field with perfect residue field, \overline{F} a fixed algebraic closure of F . For any extension E of F in \overline{F} , \mathfrak{O}_E will denote the ring of integers of E , and, if E/F has finite ramification degree, ϖ_E will denote a uniformizer in \mathfrak{O}_E and κ_E the residue field $\mathfrak{O}_E/\varpi_E\mathfrak{O}_E$. Let $p \geq 0$ denote the characteristic of κ_F and $\overline{\kappa}_F$ an algebraic closure of κ_F . Let $|\cdot|$ stand for an absolute value on F , extended uniquely to \overline{F} . Throughout, G will be a connected reductive group over a ring which, most of the time (everywhere but in §2.3), will equal F . We will let Z^0 denote the identity connected component of the center of G , G_{der} the derived group of G , G_{sc} the simply connected cover of G_{der} , and G_{ad} the adjoint group of G . Thus, we have obvious maps $G_{\mathrm{sc}} \rightarrow G \rightarrow G_{\mathrm{ad}}$. However, we will make the following exception to this convention: if $T \subset G$ is a maximal torus, T_{der} , T_{sc} and T_{ad} will denote the maximal tori of G_{der} , G_{sc} and G_{ad} determined by T . We will follow standard notation in denoting algebraic groups using upper case roman letters and their Lie algebras using the corresponding fraktur letters, for example, $\mathfrak{g}_{\mathrm{der}}$ is the Lie algebra of G_{der} .

For any extension E/F in \overline{F} of finite ramification degree, let $\mathcal{B}(G, E)$ denote the reduced Bruhat–Tits building of G over E . If E/F is Galois, then we have a canonical injection $\mathcal{B}(G, F) \rightarrow \mathcal{B}(G, E)^{\mathrm{Gal}(E/F)}$. For $x \in \mathcal{B}(G, E)$ and $r \in \mathbb{R}$, write $\mathfrak{g}(E)_{x,r} \subset \mathfrak{g}(E)$ and (when $r \geq 0$) $G(E)_{x,r} \subset G(E)$ for the corresponding Moy–Prasad lattice and subgroup, respectively.

We will omit E from all of the notation above when $E = F$, for example, ϖ will mean ϖ_F .

If X is a scheme over a ring R , and R' is an R -algebra, then $X_{R'}$ will usually denote the base change of X to R' .

If $T \subseteq G$ is a maximal torus, then $R(G, T)$ will denote the set of (absolute) roots of G with respect to T . If B is a Borel subgroup of G containing T , then $R(B, T) \subset R(G, T)$ will denote the corresponding set of positive roots, and $\Delta(B, T)$ the corresponding set of simple roots.

2.2. Topological nilpotence

In this section we take G to be a connected reductive group over F .

DEFINITION 1. Let $X \in \mathfrak{g}(F)$ and let T be an \bar{F} -torus in G such that $\mathfrak{t}(\bar{F})$ contains the semisimple part X_s of X . We say that X is *topologically nilpotent* if

$$|d\mu(X_s)| < 1 \text{ for all } \mu \in X^*(T) := \text{Hom}_{\text{alg}}(T, \mathbb{G}_m).$$

REMARK 2. (a) It is easy to see that this definition is independent of the choice of a torus T whose Lie algebra contains X_s .

(b) This definition is one of several that are commonly used. In Remark 35, we will see that it is equivalent to the one given in [4], once one has assumed that p is \mathfrak{g} -good for G and that G is *not too wild*, concepts that will be introduced later below.

NOTATION 3. Write $\mathfrak{g}(F)_{\text{tn}}$ for the set of topologically nilpotent elements of $\mathfrak{g}(F)$.

We now give a description of topological nilpotence using the adjoint quotient, by which we mean the GIT quotient $\mathfrak{g} // G$ corresponding to the adjoint action of G on \mathfrak{g} .

NOTATION 4. Write $\chi : \mathfrak{g} \rightarrow \mathfrak{q} := \mathfrak{g} // G$ for the adjoint quotient map of \mathfrak{g} over F . In future sections, we will assume this notation with F replaced by any ring R over which G is defined.

Note that any pinning of G_E , where E is a finite extension of F that splits G , determines an \mathfrak{D}_E -model $G_{\mathfrak{D}_E}$ for the base change G_E of G to E (see § 2.4 and § 2.5). Thus, \mathfrak{q}_E gets the integral model $\text{Spec } \mathfrak{D}_E[\mathfrak{g}_{\mathfrak{D}_E}]^{\mathbb{G}_{\mathfrak{D}_E}}$, which is independent of the choice of the E -pinning that defined $G_{\mathfrak{D}_E}$ (as these pinning are all $G_{\text{ad}}(E)$ -conjugate). Thus, \mathfrak{q}_E gets a canonical \mathfrak{D}_E -model for any finite extension E of F that splits G , and if two such extensions E_1 and E_2 are contained in another such extension E_3 , then the base changes of $\mathfrak{q}_{\mathfrak{D}_{E_1}}$ and $\mathfrak{q}_{\mathfrak{D}_{E_2}}$ to \mathfrak{D}_{E_3} agree as they both equal $\mathfrak{q}_{\mathfrak{D}_{E_3}}$ [29, Lemma 2]. Thus, now we may talk of $\mathfrak{q}(\mathfrak{D}_{\bar{F}})$, as well as of $\mathfrak{q}(\kappa_{\bar{F}}) = \mathfrak{q}(\bar{\kappa})$.

LEMMA 5. Let $\mathfrak{q}(1)$ denote the fiber over $\chi(0)$ of the reduction map $\mathfrak{q}(\mathfrak{D}_{\bar{F}}) \rightarrow \mathfrak{q}(\bar{\kappa})$. We have

$$\mathfrak{g}(F)_{\text{tn}} = \chi|_{\mathfrak{g}(F)}^{-1}(\mathfrak{q}(1)).$$

Proof. Intersecting with $\mathfrak{g}(F)$ yields this equality from its analogue over a finite extension E of F , so we may and do assume G to be defined and split over \mathfrak{D} .

Let us first prove that, for any $X \in \mathfrak{t}(\mathfrak{D}_{\bar{F}})$, where T is an \mathfrak{D} -split maximal torus of G , the following are equivalent:

- (i) the image of X in $\mathfrak{t}(\bar{\kappa})$ vanishes; and
- (ii) $\chi(X)$ belongs to the inverse image of $\chi(0)$ under $\mathfrak{q}(\mathfrak{D}_{\bar{F}}) \rightarrow \mathfrak{q}(\bar{\kappa})$.

It is easy to see that (i) implies (ii), so it is enough to show that if the image \bar{X} of X in $\mathfrak{t}(\bar{\kappa})$ does not vanish, then there exists a nonconstant homogeneous polynomial $f \in \mathfrak{D}[\mathfrak{g}]^G$ such that $\bar{f}(\bar{X}) \neq 0$. Choose any faithful representation $\iota : G \hookrightarrow \text{GL}(L)$ of group schemes over \mathfrak{D} , where L is a finite free \mathfrak{D} -module. The coefficients of the characteristic polynomial map on $\text{End}_{\mathfrak{D}}(L)$ determine homogeneous, nonconstant elements $f_1, \dots, f_n \in \mathfrak{D}[\mathfrak{g}]^G$, where n is the \mathfrak{D} -rank of L (we are omitting the coefficient of the highest degree term). If \bar{X} is nonzero, then $\iota(\bar{X})$ is a nonzero semisimple element of $\text{GL}(L)(\bar{\kappa})$, and hence some coefficient of its characteristic polynomial does not vanish. But this means that $\bar{f}_i(\bar{X}) \neq 0$ for some i , proving the claim.

Now let $X \in \mathfrak{g}(F)$, and let X'_s be any conjugate in $\mathfrak{t}(\overline{F})$ of the semisimple part of X . By definition, X is topologically nilpotent if and only if $|d\mu(X'_s)| < 1$ for all $\mu \in X^*(T)$, that is, if and only if X'_s both belongs to $\mathfrak{t}(\mathfrak{D}_{\overline{F}})$ and has zero as its image in $\mathfrak{t}(\kappa_{\overline{F}}) = \mathfrak{t}(\overline{\kappa})$. If $X'_s \notin \mathfrak{t}(\mathfrak{D}_{\overline{F}})$, then we can multiply X'_s by some nonunit $a \in \mathfrak{D}_{\overline{F}}$ such that aX'_s belongs to $\mathfrak{t}(\mathfrak{D}_{\overline{F}})$ and has nonzero image in $\mathfrak{t}(\overline{\kappa})$. By the proof of the equivalence of (i) and (ii) above applied to aX'_s in place of X , we see that there exists a nonconstant homogeneous polynomial $f \in \mathfrak{D}_{\overline{F}}[\mathfrak{g}]^G$ such that the image of $f(aX'_s) \in \mathfrak{D}_{\overline{F}}$ in $\overline{\kappa}$ is nonzero, that is, $f(aX'_s)$ is not divisible by a . This implies that $f(X'_s) \notin \mathfrak{D}_{\overline{F}}$, and hence $\chi(X) = \chi(X'_s) \notin \mathfrak{q}(\mathfrak{D})$, in particular $X \notin \chi|_{\mathfrak{g}(F)}^{-1}(\mathfrak{q}(1))$. Now the equality $\mathfrak{g}(F)_{\text{tn}} = \chi|_{\mathfrak{g}(F)}^{-1}(\mathfrak{q}(1))$ follows from the equivalence of (i) and (ii) above applied to $X'_s \in \mathfrak{t}(\mathfrak{D}_{\overline{F}})$ in place of X . \square

2.3. Kostant sections for split groups

In this section, we will take G to be a connected split reductive group defined over \mathbb{Z} , but will concern ourselves with its base change G_R to a ring R (commutative, with unity). Fix a pinning $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$, for G that is defined over \mathbb{Z} , with $\Delta = \Delta(B, T)$.

NOTATION 6. Let $Y = \sum_{\alpha \in \Delta} X_\alpha$. Write

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \quad (2.1)$$

for the weight space decomposition for \mathfrak{g} , over \mathbb{Z} , with respect to $\text{Ad} \circ \lambda$, where $\lambda \in X_*(T)$ is the sum of all the coroots of T in B (not just the simple ones). Let B^- be the Borel subgroup of G that is opposite to B with respect to T , with N^- as its unipotent radical.

It is easy to see that $\langle \alpha, \lambda \rangle = 2$ for all $\alpha \in \Delta$, so that $\mathfrak{g}(0) = \mathfrak{t}$ and

$$\mathfrak{b} = \bigoplus_{j \geq 0} \mathfrak{g}(j), \quad \mathfrak{b}^- = \bigoplus_{j \leq 0} \mathfrak{g}(j), \quad \mathfrak{n}^- = \bigoplus_{j < 0} \mathfrak{g}(j). \quad (2.2)$$

Further,

$$\text{for all } j \in \mathbb{Z}, \quad [Y, \mathfrak{g}(j)] \subset \mathfrak{g}(j+2). \quad (2.3)$$

For the rest of § 2.3 we assume the following statement.

HYPOTHESIS 7. The module $[Y, \mathfrak{n}_R^-]$ has a λ -invariant complement in \mathfrak{b}_R^- of rank $\text{rk}_R \mathfrak{b}_R^- - \text{rk}_R \mathfrak{n}_R^-$.

Call such a complement Ξ , and set $\mathcal{S} = Y + \Xi$.

The hypothesis is equivalent to requiring the following: for each $j < 0$, the R -submodule $[Y, \mathfrak{g}_R(j)]$ (that is, $[Y, \mathfrak{g}(j)_R]$) has a complement in $\mathfrak{g}_R(j+2)$ that is free of rank $\text{rk}_R \mathfrak{g}_R(j+2) - \text{rk}_R \mathfrak{g}_R(j)$.

Note that $[Y, \mathfrak{g}_R]$ need not have any complement in \mathfrak{g}_R even when the hypothesis is satisfied. For example, consider $G = \text{SL}_p$ over $R = \mathbb{Z}_p$.

REMARK 8. We can rephrase Hypothesis 7 as follows. Let N be the square-free natural number whose prime factors are precisely the ones occurring in the following list:

- 2 if G_{sc} has a factor of the form B_n ($n \geq 3$), D_n ($n \geq 4$), or G_2 ;
- 2 and 3 whenever G_{sc} has a factor of the form F_4 , E_6 , or E_7 ; and
- 2, 3, and 5 whenever G_{sc} has a factor of the form E_8 ;
- the primes dividing the order of $\pi_1(G_{\text{der}})$.

Note that the last condition is superfluous unless G has a factor of the form C_n or A_n .

By [32, (2.4), (2.5), and (2.6)], Hypothesis 7 is equivalent to N being invertible in R (for this amounts, in the notation of [32], to saying that the elementary divisors of t_i are all invertible in R for $i < 0$).

Thus, for instance, if $G = \mathrm{Sp}_{2n}$ ($n \geq 2$), SL_n , or GL_n , there is no restriction on R .

NOTATION 9. In the rest of this subsection we will abuse notation by confusing G , \mathfrak{b} , \mathfrak{n}^- , etc., with their base changes to R .

Note that N^- acts by the adjoint action on $Y + \mathfrak{b}^-$. We have a composite map $\mathcal{S} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g} // G$, which factors through the composite map $\mathcal{S} \hookrightarrow Y + \mathfrak{b}^- \rightarrow (Y + \mathfrak{b}^-) // N^-$.

In this section we prove, thanks to Hypothesis 7, that

PROPOSITION 10. *Both maps in the sequence $\mathcal{S} \rightarrow (Y + \mathfrak{b}^-) // N^- \rightarrow \mathfrak{g} // G$ are isomorphisms of schemes over R .*

REMARK 11. Proposition 10 says that \mathcal{S} can be viewed as a Kostant section for G over R . Thus, we have obtained a version of the Kostant section in a situation more general than that in [28]. What makes this feasible is that, unlike in [28], we do not need that the Chevalley morphism $\mathfrak{t} // W \rightarrow \mathfrak{g} // G$ is an isomorphism (which it is not under our less restrictive hypothesis, by [10, Theorem 1.2]).

The proof of the proposition is based on the following two lemmas.

LEMMA 12. *The action map $a : N^- \times \mathcal{S} \rightarrow Y + \mathfrak{b}^-$ is an isomorphism.*

Proof. The proof is adapted from that of [28, Proposition 3.2.1], which in turn is adapted from [13, Lemma 2.1]. We need to show that the map at the level of coordinate rings $a^* : R[Y + \mathfrak{b}^-] \rightarrow R[N^- \times \mathcal{S}]$ is an isomorphism.

We first prove that a^* is injective. By viewing N^- as the product of the schemes underlying its root subschemes, we can view the coordinate rings of either side as $\mathrm{Sym}^\bullet(\mathfrak{n}^{-\vee} \oplus \Xi^\vee)$ and $\mathrm{Sym}^\bullet(\mathfrak{b}^{-\vee})$ (symmetric powers of free R -modules). The map a is thus a polynomial map in several variables in R , and its linear term, as a map $\mathfrak{n}^- \oplus \Xi \rightarrow \mathfrak{b}^-$, equals:

$$a_1 : (X, Z) \mapsto [X, Y] + Z \quad (X \in \mathfrak{n}^-, Z \in \Xi).$$

This linear term is an isomorphism of affine spaces, thanks to Hypothesis 7. It now suffices to show that any polynomial self-map of an affine space over R whose degree-one term is the identity is dominant. But this follows since the induced map at the level of coordinate rings then necessarily preserves the terms of the lowest degree.

Now let us prove that a^* is surjective. The map a is equivariant for the following actions of \mathbb{G}_m : it acts by $\mathrm{Int} \circ \lambda$ on N^- and by $t \cdot x = t^{-2} \mathrm{Ad} \lambda(t)(x)$ on \mathcal{S} and $Y + \mathfrak{b}^-$. These actions of \mathbb{G}_m define decompositions of $R[N^- \times \mathcal{S}]$ and $R[Y + \mathfrak{b}^-]$ into eigenspaces.

We first show that for each $\chi \in X^*(\mathbb{G}_m) = \mathbb{Z}$, the χ -eigenspaces in $R[N^- \times \mathcal{S}]$ and $R[Y + \mathfrak{b}^-]$ are free R -modules of the same, finite, rank. Since the relevant coordinate rings are \mathbb{G}_m -equivariantly identified with $\mathrm{Sym}^\bullet(\mathfrak{n}^{-\vee} \oplus \Xi^\vee)$ and $\mathrm{Sym}^\bullet(\mathfrak{b}^{-\vee})$ (where the action on \mathfrak{n}^- is via $\mathrm{Ad} \lambda(t)$ and on Ξ and \mathfrak{b}^- via $t^{-2} \mathrm{Ad} \lambda(t)$), the assertion about eigenspaces being free modules of finite rank follows from the fact that the χ -eigenspaces in the degree-one parts of either ring are zero for $\chi \geq 0$ (the \mathbb{G}_m -actions contract the affine spaces to our base points, as $t \rightarrow \infty$). Now, to prove that the ranks of the χ -eigenspaces match, it is enough to do so degree by degree (that is, to show that for each n , the χ -eigenspaces of $\mathrm{Sym}^n(\mathfrak{n}^{-\vee} \oplus \Xi^\vee)$ and $\mathrm{Sym}^n(\mathfrak{b}^{-\vee})$

have the same rank). This reduces to the degree-one situation, which follows from a_1 being a \mathbb{G}_m -equivariant isomorphism of tangent spaces.

Now, for each $\chi \in X^*(\mathbb{G}_m)$, the restriction a_χ^* of a^* to the χ -eigenspace of $R[Y + \mathfrak{b}^-]$ is an R -linear map between two free R -modules of the same finite rank. Thus, choosing bases for these χ -eigenspaces, it is enough to show that the square matrix that represents this restriction has determinant that is a unit, that is, one that survives reduction modulo any maximal ideal of R . Since a^* respects base change, we are now reduced to assuming that R is a field. But in this case, a_χ^* , being an injective map between R -vector spaces of the same dimension, is also a surjection. \square

The following lemma does not use Hypothesis 7, and hence would be valid even if R were arbitrary.

LEMMA 13. *The map $R[\mathfrak{g}]^G \rightarrow R[Y + \mathfrak{b}^-]^{N^-}$ is injective.*

Proof. *Step 1.* We first show that the restriction map $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{b}^-]$ is injective. For this consider the conjugation map $G \times \mathfrak{b}^- \rightarrow \mathfrak{g}$. This map is G -equivariant, where we let G act on $G \times \mathfrak{b}^-$ by left translation along the first factor, and on \mathfrak{g} by the adjoint action. This map is defined over \mathbb{Z} , and it suffices to show that it is universally schematically dominant relative to \mathbb{Z} (for this would give an injection $R[\mathfrak{g}] \rightarrow R[G] \otimes_R R[\mathfrak{b}^-]$ that by G -equivariance maps $R[\mathfrak{g}]^G$ to $R[G]^G \otimes_R R[\mathfrak{b}^-] = R[\mathfrak{b}^-]$). Since \mathbb{Z} is noetherian and $G \times \mathfrak{b}^-$ is flat, by [15, Theorem 11.10.9], it suffices to prove that the morphism $G \times \mathfrak{b}^- \rightarrow \mathfrak{g}$ of R -schemes is schematically dominant when $R = k$ is a field, which we may assume to be algebraically closed. In this case, the map $G \times \mathfrak{b}^- \rightarrow \mathfrak{g}$ is surjective at the level of k -points, by [7, Proposition 14.25], giving the desired dominance as \mathfrak{g} is reduced.

Step 2. Now we show that restriction map $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{b}^-]$ factors through the map $R[\mathfrak{t}] \rightarrow R[\mathfrak{b}^-]$ that is dual to the T -equivariant projection $\mathfrak{b}^- \rightarrow \mathfrak{t}$. Thus, we need to show that the map $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{b}^-]$ equals the composite $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{t}] \rightarrow R[\mathfrak{b}^-]$ (the restriction to \mathfrak{t} , and then the T -equivariant projection). In other words, if $\varphi \in R[\mathfrak{g}]^G$, R' is an R -algebra and $X = X_0 + X_- \in \mathfrak{b}^-(R')$ with $X_0 \in \mathfrak{t}(R')$ and $X_- \in \mathfrak{n}^-(R')$, then we need to show that $\varphi(X) = \varphi(X_0)$.

Since the weights of λ^{-1} on \mathfrak{b}^- are all positive, there exists a unique morphism ι from the affine line over R' to $\mathfrak{g}_{R'}$, which on $\text{Spec } R'[t, t^{-1}]$ is given by $t \mapsto \text{Int } \lambda^{-1}(t)(X_0 + X_-)$, and takes 0 to X_0 . Since φ is G -invariant, $\varphi \circ \iota$ is a constant on $\text{Spec } R'[t, t^{-1}]$, and hence on the entire affine line over R' . This gives

$$\varphi(X_0) = \varphi \circ \iota(0) = \varphi \circ \iota(1) = \varphi(X),$$

as needed.

Step 3. By Steps 1 and 2, the restriction map $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{t}]$ is injective. Applying Step 2, this time with \mathfrak{b} in place of \mathfrak{b}^- , the restriction maps $R[\mathfrak{g}]^G \rightarrow R[\mathfrak{t}]$ and $R[\mathfrak{g}]^G \rightarrow R[Y + \mathfrak{t}]$ coincide, once the ‘translation by Y ’ identification of \mathfrak{t} with $Y + \mathfrak{t}$ is made. Thus, the restriction map $R[\mathfrak{g}]^G \rightarrow R[Y + \mathfrak{t}]$ is injective, and *a fortiori*, the map $R[\mathfrak{g}]^G \rightarrow R[Y + \mathfrak{b}^-]$ is injective. \square

REMARK 14. The proof of Lemma 13 yields a shorter proof of the result proved in [10] that the Chevalley morphism $\mathfrak{t} // W \rightarrow \mathfrak{g} // G$ over an arbitrary scheme is dominant whenever G is almost simple (a restriction we see is no longer necessary).

We now prove Proposition 10.

Proof of Proposition 10. Since $R[\mathcal{S}]$ is free over R and the map

$$R[Y + \mathfrak{b}^-] \rightarrow R[N^- \times \mathcal{S}] = R[N^-] \otimes_R R[\mathcal{S}]$$

is N^- -equivariant and an isomorphism (Lemma 12), it follows that the map $R[Y + \mathfrak{b}^-]^{N^-} \rightarrow R[\mathcal{S}]$ is an isomorphism too. Therefore, it is enough to show that the map $R[\mathfrak{g}]^G \rightarrow R[Y + \mathfrak{b}^-]^{N^-}$, which we already know to be an injection, is also surjective. This assertion is independent of \mathcal{S} , so we may choose any $\mathcal{S} = Y + \Xi$ we like. We choose Ξ to be of the form $\dot{\Xi}_R$, where $\dot{\Xi} \subset \mathfrak{b}^-(\mathbb{Z}[1/N])$ is a λ -invariant $\mathbb{Z}[1/N]$ -submodule complementary to $[Y, \mathfrak{n}^-(\mathbb{Z}[1/N])]$. It suffices to show that the composite map $R[\mathfrak{g}]^G \hookrightarrow R[Y + \mathfrak{b}^-]^{N^-} \rightarrow R[\mathcal{S}]$ is surjective. Considering the chain

$$\mathbb{Z}[1/N][\mathfrak{g}]^G \otimes_{\mathbb{Z}[1/N]} R \rightarrow R[\mathfrak{g}]^G \hookrightarrow R[Y + \mathfrak{b}^-]^{N^-} \rightarrow R[\mathcal{S}] = \mathbb{Z}[1/N][\mathcal{S}] \otimes_{\mathbb{Z}[1/N]} R,$$

it is enough to prove the assertion of the proposition for $R = \mathbb{Z}[1/N]$. We now follow the proof of [10, Corollary 3.7]. $\mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[\mathcal{S}]$ is an isomorphism, and hence (using faithful flatness and [29, Lemma 2]) so is $\mathbb{Q}[\mathfrak{g}]^G \rightarrow \mathbb{Q}[\mathcal{S}]$. Given $P \in \mathbb{Z}[1/N][\mathcal{S}]$, one can therefore find $Q \in \mathbb{Q}[\mathfrak{g}]^G$ with image P . Write $Q = (r/s)Q_0$, where Q_0 is a primitive polynomial in $\mathbb{Z}[1/N][\mathfrak{g}]$, and r and s are coprime in $\mathbb{Z}[1/N]$. Now if some prime p that does not divide N divides s , then the image of rQ_0 in $(\mathbb{Z}/p\mathbb{Z})[\mathcal{S}]$ is zero. But this means, by the injectivity of $(\mathbb{Z}/p\mathbb{Z})[\mathfrak{g}]^G \hookrightarrow (\mathbb{Z}/p\mathbb{Z})[\mathcal{S}]$, that $rQ_0 = 0$ in $(\mathbb{Z}/p\mathbb{Z})[\mathfrak{g}]$, a contradiction. \square

2.4. Pinnings, regular nilpotent elements, and hyperspecial points

From now on, we assume that G is a quasi-split reductive group over F . Let E be a finite Galois extension of F in \bar{F} that splits some (hence any) maximal F -torus in a Borel F -subgroup of G . Our goal in this subsection is to recall the following maps, both of which are equivariant under the actions of $G(F)$:

$$\begin{array}{ccc} \{F\text{-pinnings of } G\} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \begin{array}{c} \{\text{regular nilpotent elements of } \mathfrak{g}(F)\} \\ \{E\text{-hyperspecial points in } \mathcal{B}(G)\} \end{array} \end{array} \quad (2.4)$$

Here, by an F -pinning $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ of G , we mean that (B, T) is a Borel-torus pair for G , defined over F , $\Delta = \Delta(B, T)$, each X_α is a nonzero vector in $\mathfrak{g}_\alpha(E)$, where \mathfrak{g}_α is the α -eigenspace for the action of T_E on \mathfrak{g}_E , and the set $\{X_\alpha\}$ is stable under $\text{Gal}(E/F)$. Note that we automatically have $X_\alpha \in \mathfrak{g}(E')$ for any extension E' of F in \bar{F} that splits G (and hence T). Thus the definition of an F -pinning is independent of the choice of E .

Then the upper arrow in (2.4) sends an F -pinning $(B, T, \{X_\alpha\})$ to the element $\sum_{\alpha \in \Delta} X_\alpha$ of $\mathfrak{g}(F)$, which is regular nilpotent (see [28, Lemma 3.1.1]). This map is clearly $G_{\text{ad}}(F)$ -equivariant.

When F has characteristic zero, it is well known that this map induces a bijection at the level of $G(F)$ -conjugacy classes (cf. [23, §5.1]). Even though we do not use it, let us mention that similar methods work in general when p is \mathfrak{g} -good for G , a notion that we will introduce in Definition 15.

Now we will use our pinning $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ to determine a point $x \in \mathcal{B}(G)$ that is hyperspecial over E . First, our pinning can be extended to a Chevalley system. Bruhat-Tits [9, §4] associates to such a system a valuation of the root groups for G over E . Thus from [8, §6.2], we obtain a hyperspecial point x in the apartment $\mathcal{A}(T, E)$ of T in the building $\mathcal{B}(G, E)$, independent of the choice of the Chevalley system extending our fixed pinning. Since our pinning is invariant under $\text{Gal}(E/F)$, we have that $x \in \mathcal{A}(T, E)^{\text{Gal}(E/F)} = \mathcal{A}(T, F) \subseteq \mathcal{B}(G)$ (here $\mathcal{A}(T, F)$ means $\mathcal{A}(S, F)$, where S is the maximal F -split subtorus of T). It is easy to see

that x is independent of the choice of E . Henceforth, we will be working with our fixed pinning, but the field E will not be fixed.

2.5. The setup

We continue to assume that G is quasi-split over F , and from now on fix B , T , $\{X_\alpha\}_{\alpha \in \Delta}$, B^- , N^- , λ , Y , and hence obtain $\mathfrak{g}(j)$ as in Notation 6, except that these objects are now all defined over F . Starting from $(B, T, \{X_\alpha\})$, the maps of (2.4) give a regular nilpotent element of $\mathfrak{g}(F)$, which coincides with Y , and a point $x \in \mathcal{B}(G)$ that becomes hyperspecial over any finite extension E of F that splits T .

Thus, for any such E , x determines a model for G_E over \mathfrak{O}_E , which is also the \mathfrak{O}_E -model defined by a choice of a Chevalley basis over E associated to the pinning $(B, T, \{X_\alpha\})$.

We will abuse notation by letting G also stand for this model. This will not create any confusion, as $G(R)$ will still have a well-defined meaning when R is both an F -algebra and an \mathfrak{O}_E -algebra. Similarly, we will use T to also denote its obvious model over \mathfrak{O}_E arising from any identification of its base change to E with a product of copies of \mathbb{G}_m over E .

2.6. Conditions on the residual characteristic

We are interested in conditions on p that ensure that the adjoint action induced by Y on \mathfrak{g}_{κ_E} is smoothly regular. For some purposes, we only require something weaker.

DEFINITION 15. (a) Say that p is \mathfrak{n}^- -good (for G) if the restriction of $\text{ad } Y$ from the κ_E -vector space $\mathfrak{g}(\kappa_E)$ to $\mathfrak{n}^-(\kappa_E)$ has rank equal to $\dim_F \mathfrak{n}^-$. Equivalently, $[Y, \mathfrak{n}_{\mathfrak{O}_E}^-]$ has a λ -invariant complement in $\mathfrak{b}_{\mathfrak{O}_E}^-$ of rank $\dim_F \mathfrak{b}^- - \dim_F \mathfrak{n}^- (= \text{rk } G)$.

(b) Say that p is \mathfrak{g} - F -good (for G) if for every real number r , the map $\overline{\text{ad } Y} : \mathfrak{g}_{x,r}(j)/\mathfrak{g}_{x,r+}(j) \rightarrow \mathfrak{g}_{x,r}(j+2)/\mathfrak{g}_{x,r+}(j+2)$ induced by $\text{ad } Y$ is injective for every $j < 0$ and surjective for $j \geq 0$.

(c) Say that p is \mathfrak{g} -good (for G) if p is \mathfrak{g} - E -good for G . Equivalently, the endomorphism $\text{ad } Y$ on the κ_E -vector space $\mathfrak{g}(\kappa_E)$ has rank equal to $\dim G - \text{rk } G$. Equivalently, $[Y, \mathfrak{g}_{\mathfrak{O}_E}]$ has a λ -invariant complement in $\mathfrak{g}_{\mathfrak{O}_E}$ of rank equal to $\text{rk } G$.

We will see in Corollary 26 how the conditions of \mathfrak{g} -good and \mathfrak{g} - F -good compare. In particular, they are equivalent for tamely ramified groups G .

REMARK 16. Whether p is \mathfrak{g} -good (or \mathfrak{n}^- -good) for G depends only on the absolute root system for G unless G has a factor of type A_n or C_n , in which case it depends on the absolute root datum.

(a) We have that p is \mathfrak{n}^- -good if and only if G satisfies Hypothesis 7 with $R = \mathfrak{O}_E$. Therefore, Remark 8 describes precisely which values of p are \mathfrak{n}^- -good.

(b) From [36, Theorem 52] (and earlier [32, Theorem 5.9] in the case where G is semisimple), p is \mathfrak{g} -good if and only if

- p is \mathfrak{n}^- -good;
- $p \neq 2$ (respectively, 3) if G has a factor of type C_n (respectively, G_2); and
- p does not divide the order of $\pi_1(G_{\text{der}})$ or the order (in the scheme-theoretic sense) of the component group of the center.

Note that this last condition is superfluous unless G has a factor of type A_n , and all values of $p \geq 0$ are \mathfrak{g} -good for a general linear group.

Rather than assuming that G splits over a tamely ramified extension of F , we will sometimes be interested in situations where the following weaker condition is met.

DEFINITION 17. Say that G is *not too wild* if it satisfies the following.

- (i) Write $G_{\text{sc}} = \prod_i \text{Res}_{E_i/F} H_i$, where each H_i is an absolutely almost simple, simply connected group over a finite, separable extension E_i of F . Then each group H_i is split over a tamely ramified extension of E_i .
- (ii) Whenever E' is any extension that splits G , then $\mathfrak{t}_{\text{der}}(\mathfrak{D}_{E'})$ has a $\text{Gal}(E'/F)$ -invariant complement in $\mathfrak{t}(\mathfrak{D}_{E'})$.

Even though we will not use it, we mention in passing that it is easy to check from the proof of Lemma 22 that if p is \mathfrak{n}^- -good and E'/F can be chosen to be tamely ramified, then part (ii) of the definition holds, hence G is not too wild.

If p is \mathfrak{n}^- -good, then G satisfies part (i) of the definition except possibly if $p = 2$ and some H_i is of type A_n ($n > 1$), or if $p = 3$ and some H_i is a triality form of D_4 . Part (ii) of course needs to be verified only over the minimal extension of F in \overline{F} that splits T , and is automatic when G is semisimple.

REMARK 18. Here is an example illustrating that Condition (ii) in Definition 17 does not follow from Condition (i) of the same definition. Let $F = \mathbb{Q}_2$, $E = \mathbb{Q}_2[\sqrt{2}]$. Note that there is an embedding $\mu_2 \hookrightarrow \text{Res}_{E/F} \mathbb{G}_m$, which at the level of R -points is given by $a \mapsto 1 \otimes a \in (E \otimes_F R)^\times = \text{Res}_{E/F} \mathbb{G}_m(R)$. μ_2 also embeds into SL_2 as its center. Let $G = (\text{Res}_{E/F} \mathbb{G}_m \times \text{SL}_2) / \Delta(\mu_2)$, where Δ stands for the diagonal embedding. Since SL_2 is split, Condition (i) of Definition 17 is automatically satisfied. We claim that Condition (ii) is not. The lattice $X_*(\text{Res}_{E/F} \mathbb{G}_m)$ has a basis $\{e'_2, e'_3\}$ permuted by $\text{Gal}(E/F)$, and let e'_1 denote a basis element for the cocharacter lattice of the standard maximal torus T_{sc} of SL_2 . Then we have an identification:

$$X_*(T) = \{(a, b, c) \in ((1/2)\mathbb{Z})^3 \mid a \equiv b \equiv c \pmod{\mathbb{Z}}\},$$

so that $X_*(T)$ is spanned by e_1, e_2, e_3 , where $e_1 := e'_1$, $e_2 := (1/2)(e'_1 + e'_2 + e'_3)$ and $e_3 := e'_3$. These also give a basis for $\mathfrak{t}(E)_0 = X_*(T) \otimes_{\mathbb{Z}} \mathfrak{D}_E$. We claim that $\mathfrak{t}_{\text{sc}}(E)_0 = \mathfrak{D}_E e_1$ does not have a $\text{Gal}(E/F)$ -invariant complement in $\mathfrak{t}(E)_0$. To see this, we pass to $\kappa_E = \kappa_F$, upon which the nontrivial element σ of $\text{Gal}(E/F)$ becomes a unipotent matrix T_σ that fixes the images \bar{e}_1 and \bar{e}_2 of e_1 and e_2 , and takes the image \bar{e}_3 of e_3 to $\bar{e}_1 + \bar{e}_3$. We conclude that $T_\sigma - 1$ has \bar{e}_1 in its image, which would have been impossible if $\kappa \bar{e}_1$ had a T_σ -invariant complement.

2.7. Kostant sections over F and \mathfrak{D}_E

Until the end of the proof of Lemma 22, fix a finite Galois extension E over which G splits. Let K be the subextension of E such that E/K is totally ramified and K/F is unramified. Recall that we will sometimes view G , B , T , etc., also as groups defined over \mathfrak{D}_E , using the Chevalley basis that we have fixed in §2.5. For every $r \in \mathbb{R}$, we have $G(F)$ -invariant subsets \mathfrak{g}_r (respectively, \mathfrak{g}_{r+}) in $\mathfrak{g}(F)$, defined as the union of Moy–Prasad filtration sublattices $\mathfrak{g}_{y,r}$ (respectively, $\mathfrak{g}_{y,r+}$) as y ranges over $\mathcal{B}(G)$. These filtration sublattices are normalized as in [2, §2.1.2], or equivalently as in [4, §1.4], unlike in [26], so that $\mathfrak{g}_{y,r+1} = \varpi \mathfrak{g}_{y,r}$ for all $y \in \mathcal{B}(G)$. Let e be the ramification degree of E/F , so that $[E : K] = e$. Choose $0 = r_0 < r_1 < \dots < r_m = 1$ such that

$$\mathfrak{g}_{x,r_i} \supsetneq \mathfrak{g}_{x,r_i+} = \mathfrak{g}_{x,r_{i+1}}, \quad \text{for all } 0 \leq i < m. \quad (2.5)$$

REMARK 19. (i) If G is not too wild, then for all $r \in \mathbb{R}$, $\mathfrak{g}_{x,r} = \mathfrak{g}(E)_{x,re} \cap \mathfrak{g}(F)$ and $\mathfrak{g}(K)_{x,r} = \mathfrak{g}(E)_{x,re} \cap \mathfrak{g}(K)$ (see [1, Proposition 1.4.1; 6, Lemma 3.14]). We remark in passing that this assertion does not need the full strength of G being not too wild: If we write $G_{\text{sc}} = \prod_i \text{Res}_{E_i/F} H_i$ as in Definition 17(i), then it suffices to assume that each H_i is not a

wild special unitary group (that is, either H_i is not a special unitary group or it splits over a tamely ramified extension). Since $\text{Gal}(E/F)$ fixes the point x , it also preserves $\mathfrak{g}(E)_{x,r}$ for all r .

(ii) Since x is hyperspecial over E , if G is not too wild, then (i) gives that for each i , $r_i = j_i/e$ for some $j_i \in \{0, \dots, e\}$.

(iii) For all $r \in \mathbb{R}$, since $\mathfrak{g}_{x,r} = \mathfrak{g}(K)_{x,r} \cap \mathfrak{g}(F)$, étale descent gives that the obvious map $\mathfrak{g}_{x,r} \otimes_{\mathfrak{O}_K} \mathfrak{O}_K \rightarrow \mathfrak{g}(K)_{x,r}$ is an isomorphism. Therefore, the analogue of (2.5) over K holds with the same numbers r_i , so that

$$\mathfrak{g}(K)_{x,r_i} \supsetneq \mathfrak{g}(K)_{x,r_i+} = \mathfrak{g}(K)_{x,r_{i+1}}, \quad \forall 0 \leq i < m. \quad (2.6)$$

Recall that $Y = \sum_{\alpha \in \Delta(B,T)} X_\alpha \in \mathfrak{g}(F)$ is regular nilpotent, and belongs to $\mathfrak{g}(E)_{x,0} \cap \mathfrak{g}(F) = \mathfrak{g}_{x,0}$. Assuming G to be not too wild, we have a map $\mathfrak{g}(K)_{x,r_i} \rightarrow \mathfrak{g}(E)_{x,0}$ given by $X \mapsto \varpi_E^{-j_i} X$. This induces a $\kappa_K = \kappa_E$ -linear map

$$\xi_i: \mathfrak{g}(K)_{x,r_i}/\mathfrak{g}(K)_{x,r_i+} \rightarrow \mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+}$$

that is equivariant for the map induced on either side by $\text{ad } Y$ (which makes sense since $[\mathfrak{g}(K)_{x,r}, \mathfrak{g}(K)_{x,s}] \subset \mathfrak{g}(K)_{x,r+s}$ and $[\mathfrak{g}(E)_{x,r}, \mathfrak{g}(E)_{x,s}] \subset \mathfrak{g}(E)_{x,r+s}$ for all $r, s \in \mathbb{R}$).

NOTATION 20. For a subset $L \subset \mathfrak{g}(\overline{F})$, write $L(j)$ for $L \cap \mathfrak{g}(j)(\overline{F})$ (see Notation 6).

From equation (2.1) (recall that x is hyperspecial over E) we get that for all $r \in \mathbb{R}$,

$$\mathfrak{g}(E)_{x,r} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(E)_{x,r}(j).$$

Since $\mathfrak{g}_{x,r}$ is a sum of \mathfrak{O} -submodules of eigenspaces for the adjoint action of the maximal F -split subtorus of T , and similarly with $\mathfrak{g}(K)_{x,r}$, we have

$$\mathfrak{g}(K)_{x,r} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(K)_{x,r}(j), \quad \text{and} \quad \mathfrak{g}_{x,r} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{x,r}(j). \quad (2.7)$$

REMARK 21. Equation (2.7), together with the fact that $\varpi \mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$ for all $r \in \mathbb{R}$, allows us to rephrase the notion of \mathfrak{g} - F -good as follows. p is \mathfrak{g} - F -good if and only if the rank of the κ -vector space endomorphism:

$$\overline{\text{ad } Y}: \bigoplus_{\substack{0 \leq r < 1 \\ \mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r+}}} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \rightarrow \bigoplus_{\substack{0 \leq r < 1 \\ \mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r+}}} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$$

induced by $\text{ad } Y$ equals

$$\begin{aligned} \sum_{j \neq 0} \sum_{\substack{0 \leq r < 1 \\ \mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r+}}} \dim_{\kappa} \mathfrak{g}_{x,r}(j)/\mathfrak{g}_{x,r+}(j) &= \sum_{j \neq 0} \dim_{\kappa} \mathfrak{g}_{x,0}(j)/\mathfrak{g}_{x,1}(j) \\ &= \sum_{j \neq 0} \text{rk}_{\mathfrak{O}} \mathfrak{g}_{x,0}(j) = \sum_{j \neq 0} \dim_F \mathfrak{g}(F)(j), \end{aligned}$$

which equals $\dim G - \text{rk } G$.

LEMMA 22. If G is not too wild, each ξ_i is injective. Moreover, if E/F is tamely ramified, the map

$$\xi := \bigoplus_{i=0}^{m-1} \xi_i: \bigoplus_{i=0}^{m-1} \mathfrak{g}(K)_{x,r_i}/\mathfrak{g}(K)_{x,r_i+} \rightarrow \mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+} \quad (2.8)$$

is an isomorphism of vector spaces over $\kappa_K = \kappa_E$ that is equivariant for the action induced on both sides by $\text{ad } Y$.

Proof. Since G is not too wild, Remark 19(i) applies. The map ξ_i is injective because if $X \in \mathfrak{g}(K)_{x,r_i}$ satisfies $\varpi_E^{-j_i} X \in \mathfrak{g}(E)_{x,\varepsilon}$ for some $\varepsilon > 0$, then

$$X \in \varpi_E^{j_i} \mathfrak{g}(E)_{x,\varepsilon} \cap \mathfrak{g}(K) = \mathfrak{g}(E)_{x,j_i+\varepsilon} \cap \mathfrak{g}(K) = \mathfrak{g}(K)_{x,(j_i/e)+(\varepsilon/e)} \subset \mathfrak{g}(K)_{x,r_i+}.$$

The domain and codomain of ξ have the same dimension, namely $\dim G$. Now it suffices to show the linear disjointness of the images of the ξ_i . It is easy to see that each $\sigma \in \text{Gal}(E/K)$ induces a κ_K -linear transformation on $\mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+}$ that acts by $(\varpi_E \sigma(\varpi_E)^{-1})^{j_i}$ on the image of ξ_i . Since the characters $\sigma \mapsto (\varpi_E \sigma(\varpi_E)^{-1})^j$ of $\text{Gal}(E/K)$, $0 \leq j < e$, are all distinct, the linear disjointness of the images of the ξ_i follows, and hence so does the lemma. \square

REMARK 23. If E/F is wildly ramified, then ξ need not be an isomorphism. For example, consider the case when $G = \text{Res}_{\mathbb{Q}_2[\sqrt{2}]/\mathbb{Q}_2} \mathbb{G}_m$.

LEMMA 24. Suppose E is any Galois extension of F splitting G . Give G an \mathfrak{D}_E -structure using our fixed pinning. Then the following are equivalent.

- (I) There exists a finitely generated $\text{Gal}(E/F)$ -invariant \mathfrak{D}_E -submodule $L_{\mathfrak{D}_E} \subset \mathfrak{g}(\mathfrak{D}_E)$ such that $L_{\mathfrak{D}_E} = \bigoplus_{j \in \mathbb{Z}} L_{\mathfrak{D}_E}(j)$ is a λ -invariant complement in $\mathfrak{b}^-(\mathfrak{D}_E)$ for $[Y, \mathfrak{n}^-(\mathfrak{D}_E)]$ of rank $\text{rk } G$.
- (II) G is not too wild and p is \mathfrak{n}^- -good.

If the equivalent properties (I) and (II) are satisfied, then for any lattice $L_{\mathfrak{D}_E}$ as in (I), the following additional properties are equivalent:

- (III) $\mathfrak{g}(E)_{x,0} = [Y, \mathfrak{g}(E)_{x,0}] + L_{\mathfrak{D}_E} + \mathfrak{g}(E)_{x,0+}$.
- (IV) p is \mathfrak{g} -good.

REMARK 25. The validity of Condition (I) of Lemma 24 does not depend on the choice of E . To see this, let $E \subset E'$ be an inclusion of finite Galois extensions of F in \bar{F} splitting G . Our pinning gives models $G_{\mathfrak{D}_E}$ and $G_{\mathfrak{D}_{E'}}$ for G over \mathfrak{D}_E and $\mathfrak{D}_{E'}$. Given a lattice $L_{\mathfrak{D}_E}$ satisfying Condition (I), note that $L_{\mathfrak{D}_E} \otimes_{\mathfrak{D}_E} \mathfrak{D}_{E'}$ satisfies the analogous condition over E' . Conversely, given a lattice $L_{\mathfrak{D}_{E'}}$ that satisfies the analogue of Condition (I) over E' , the lattice $L_{\mathfrak{D}_{E'}}^{\text{Gal}(E'/E)}$ satisfies Condition (I). Similarly, the validity of Condition (III) does not depend on the choice of E either.

Proof of Lemma 24. Proof of (I) being equivalent to (II): Suppose for this paragraph that (I) holds. Then p is \mathfrak{n}^- -good (see the second sentence of Definition 15(a)). Moreover, p being \mathfrak{n}^- -good forces $[Y, \mathfrak{n}^-(\kappa)] \cap \mathfrak{t}(\kappa) = \mathfrak{t}_{\text{der}}(\kappa)$ by dimension considerations, hence $[Y, \mathfrak{n}^-(\mathfrak{D}_E)] \cap \mathfrak{t}(\mathfrak{D}_E) = \mathfrak{t}_{\text{der}}(\mathfrak{D}_E)$ by Nakayama's lemma, so that $L_{\mathfrak{D}_E}$ furnishes a $\text{Gal}(E/F)$ -invariant complement to $\mathfrak{t}_{\text{der}}(\mathfrak{D}_E)$ in $\mathfrak{t}(\mathfrak{D}_E)$. Thanks to Remark 25, this means that Condition 17(ii) holds.

Thus, we now suspend the assumption that (I) holds, and assume these two conditions (p being \mathfrak{n}^- -good and Condition 17(ii) holding) for the remainder of the proof, because otherwise neither (I) nor (II) is satisfied. We present the remainder of the proof in three steps.

Step 1. Proof in the tamely ramified case. Suppose G splits over a tamely ramified extension, which may and will be taken to be Galois over F . By Remark 25, we may take E/F to be such an extension.

In this case, Condition 17(i) is automatic, and we need to construct $L_{\mathfrak{D}_E} = \bigoplus_{j \leq 0} L_{\mathfrak{D}_E}(j)$ satisfying Property (I).

We let $L_{\mathfrak{D}_E}(0)$ be a $\text{Gal}(E/F)$ -equivariant complement of $\mathfrak{t}_{\text{der}}(\mathfrak{D}_E)$ in $\mathfrak{t}(\mathfrak{D}_E)$, which exists by Condition 17(ii).

Next let $j < 0$. Let $i \in \{0, \dots, m-1\}$. Then \mathfrak{g}_{x,r_i} and \mathfrak{g}_{x,r_i+} decompose as $\bigoplus_{\ell} \mathfrak{g}_{x,r_i}(\ell)$ and $\bigoplus_{\ell} \mathfrak{g}_{x,r_i+}(\ell)$, compatibly. Consider the map

$$\mathfrak{g}_{x,r_i}(j-2)/\mathfrak{g}_{x,r_i+}(j-2) \rightarrow \mathfrak{g}_{x,r_i}(j)/\mathfrak{g}_{x,r_i+}(j) \quad (2.9)$$

induced by $\text{ad } Y$. Choose a κ -basis for a complement of the image of this map, and lift it to a subset $A_{i,j} \subset \mathfrak{g}_{x,r_i}(j)$. Take $L_{\mathfrak{D}_E}(j)$ to be the \mathfrak{D}_E -span of $\varpi_E^{-ji} A_{i,j}$ ($i \in \{0, \dots, m-1\}$). It is clearly a $\text{Gal}(E/F)$ -invariant, finitely generated \mathfrak{D}_E -submodule of $\mathfrak{g}(E)_{x,0}$. From Remark 19(iii), Lemma 22, and the fact that $\dim_{\kappa_E}[Y, \mathfrak{g}(j)(\kappa_E)] = \dim_E[Y, \mathfrak{g}(E)(j)]$ (which follows since p is \mathfrak{n}^- -good), it follows that $L_{\mathfrak{D}_E}(j)$ is indeed a complement for $[Y, \mathfrak{g}(E)_{x,0}(j-2)]$ in $\mathfrak{g}(E)_{x,0}(j)$. Hence $L_{\mathfrak{D}_E} = \bigoplus_{j \leq 0} L_{\mathfrak{D}_E}(j)$ is a λ -invariant complement in $\mathfrak{b}^-(\mathfrak{D}_E)$ for $[Y, \mathfrak{n}^-(\mathfrak{D}_E)]$.

Step 2: Proof when Condition 17(i) is satisfied. We need to show the existence of $L_{\mathfrak{D}_E} = \bigoplus_{j \leq 0} L_{\mathfrak{D}_E}(j)$ as before. Again, we let $L_{\mathfrak{D}_E}(0)$ be a $\text{Gal}(E/F)$ -equivariant complement of $\mathfrak{t}_{\text{der}}(\mathfrak{D}_E)$ in $\mathfrak{t}(\mathfrak{D}_E)$, which exists by Condition 17(ii).

In order to define $L_{\mathfrak{D}_E}(j)$ for $j < 0$, recall that we write $G_{\text{sc}} = \prod_i \text{Res}_{E'_i/F} H_i$, where each H_i splits over a tamely ramified extension of E'_i by Condition 17(i). Thus, as a $\text{Gal}(E/F)$ - \mathfrak{D}_E -module, we have for $j \neq 0$:

$$\mathfrak{g}(E)(j) \simeq \bigoplus_i \bigoplus_{\sigma \in \mathcal{Y}_i} \sigma \cdot \mathfrak{h}_i(E)(j),$$

and $Y = \sum_i \sum_{\sigma \in \mathcal{Y}_i} \sigma \cdot Y_i$, where $Y_i = \sum_{\alpha \in \Delta_i} X_{\alpha}$ with Δ_i being a basis of the roots of H_i compatible with Δ , and \mathcal{Y}_i denotes a set of representatives of $\text{Gal}(E/F)/\text{Gal}(E/E'_i)$.

By Step 1, for each i , there exists a lattice $L_{H_i, \mathfrak{D}_{E'_i}}(j) \subset \mathfrak{h}_i(E)(j)$ that is a $\text{Gal}(E/E'_i)$ -invariant complement of $[Y_i, \mathfrak{h}_i(E)(j-2)]$. Define

$$L_{\mathfrak{D}_E}(j) := \bigoplus_i \bigoplus_{\sigma \in \mathcal{Y}_i} \sigma \cdot L_{H_i, \mathfrak{D}_{E'_i}}(j).$$

Let $L_{\mathfrak{D}_E} := \bigoplus_{j \leq 0} L_{\mathfrak{D}_E}(j)$. Then $L_{\mathfrak{D}_E}$ is $\text{Gal}(E/F)$ -invariant and satisfies (I) by construction.

Step 3: Proof when Condition 17(i) is not satisfied. There are two situations to treat here:

- one of the groups H_i is of type A_n , splitting over only a ramified quadratic extension (in particular, $n \geq 2$), and $p = 2$; and
- one of the groups H_i is of type D_4 , on which the action of the absolute Galois group of E'_i has order at least three, and $p = 3$.

We consider the case $G = H_1$ first and deduce the more general case afterwards.

Let us consider the A_n case first, split over a ramified quadratic extension E over F . In this case, $n \geq 2$, $\mathfrak{g}(\mathfrak{D}_E)(-2n)$ is of rank 1, and $\mathfrak{g}(\mathfrak{D}_E)(-2n+2)$ is of rank 2. Further, $\mathfrak{g}(\mathfrak{D}_E)(-2n+2)$ is the direct sum of $[Y, \mathfrak{g}(\mathfrak{D}_E)(-2n)]$ and another rank-one \mathfrak{D}_E -submodule (all the assertions so far are a GL_n -computation or a computation using a Chevalley system, depending on the reader's preference). Thus, analogous results hold over κ_E , too. The group $\text{Gal}(E/F)$ exchanges the root spaces spanning $\mathfrak{g}(\mathfrak{D}_E)(-2n+2)$, as it exchanges the corresponding roots and induces an automorphism of $\mathfrak{g}(E)_{x,0}$. Thus, we find that the κ_E -vector space $\mathfrak{g}(\kappa_E)(-2n+2)$ is spanned by two vectors e_1, e_2 , which are exchanged by the κ_E -linear automorphism T_{σ} induced by the unique nontrivial element $\sigma \in \text{Gal}(E/F)$. Since $p = 2$, T_{σ} is unipotent. If $[Y, \mathfrak{g}(\mathfrak{D}_E)(-2n)]$ had a $\text{Gal}(E/F)$ -invariant complement in $\mathfrak{g}(\mathfrak{D}_E)(-2n+2)$, then $[\bar{Y}, \mathfrak{g}(\kappa_E)(-2n)]$ would have a T_{σ} -invariant complement in $\mathfrak{g}(\kappa_E)(-2n+2)$. But $[\bar{Y}, \mathfrak{g}(\kappa_E)(-2n)]$ is a one-dimensional κ_E -vector space (since p is \mathfrak{n}^- -good, it is enough to check the analogous result for a general linear group

over \mathbb{C}). Thus, the existence of a T_σ -invariant complement would imply that T_σ was a nontrivial semisimple automorphism of $\mathfrak{g}(\kappa_E)(-2n+2)$, a contradiction.

Now we consider the D_4 case. Here the argument is somewhat similar. We consider $[Y, \mathfrak{g}(\mathfrak{D}_E)(-8)] \subset \mathfrak{g}(\mathfrak{D}_E)(-6)$. If $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the simple roots of D_4 , with α_2 the unique root fixed by $\text{Gal}(\overline{F}/F)$, then $\mathfrak{g}(\mathfrak{D}_E)(-8)$ is spanned by the root space corresponding to $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, while $\mathfrak{g}(\mathfrak{D}_E)(-6)$ is spanned by the root spaces corresponding to the $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \alpha_i$ as i ranges over 1, 3, and 4. $[Y, \mathfrak{g}(\mathfrak{D}_E)(-8)]$ is a rank one \mathfrak{D}_E -submodule of $\mathfrak{g}(\mathfrak{D}_E)(-6)$, admitting a complementary rank two \mathfrak{D}_E -submodule (use that p is \mathfrak{n}^- -good). As before, we have an automorphism T_σ of $\mathfrak{g}(\kappa_E)(-6)$, which permutes three basis vectors in a three cycle and hence can be checked to have minimal polynomial $(T-1)^3$ (thanks to $p=3$). But if the image of $[Y, \mathfrak{g}(\mathfrak{D}_E)(-8)]$ in $\mathfrak{g}(\kappa_E)$ had a T_σ -invariant complement in $\mathfrak{g}(\kappa_E)(-6)$, then the minimal polynomial of T_σ on $\mathfrak{g}(\kappa_E)(-6)$ would have to be $(T-1)^2$, a contradiction.

Now drop our assumption that $G = H_1$, and write $G_{\text{sc}} = \prod_i \text{Res}_{E'_i/F} H_i$, where H_1 is of one of the two types mentioned at the beginning of Step 3, and $p=2$ or $p=3$ as appropriate. Let $j = -2n+2$ if $p=2$, and $j = -6$ if $p=3$. To get a contradiction, assume that for some splitting field E of F , $[Y, \mathfrak{n}^-(\mathfrak{D}_E)]$ has a $\text{Gal}(E/F)$ -invariant λ -invariant complement in $\mathfrak{b}^-(\mathfrak{D}_E)$, that is, in particular, $[Y, \mathfrak{g}(\mathfrak{D}_E)(j-2)]$ has a $\text{Gal}(E/F)$ -invariant complement in $\mathfrak{g}(\mathfrak{D}_E)(j)$. Recall that $\mathfrak{g}(\mathfrak{D}_E)(j) \simeq \bigoplus_i \bigoplus_{\sigma \in \mathcal{Y}_i} \sigma \cdot \mathfrak{h}_i(\mathfrak{D}_E)(j)$, and $\text{Gal}(E/F)$ preserves each summand $\bigoplus_{\sigma \in \mathcal{Y}_i} \sigma \cdot \mathfrak{h}_i(\mathfrak{D}_E)(j)$. Hence, using the notation from Step 2, we obtain a $\text{Gal}(E/F)$ -invariant complement of the image of $\text{ad}(\sum_{\sigma \in \mathcal{Y}_i} \sigma \cdot Y_i)$ in $\bigoplus_{\mathcal{Y}_i} \sigma \cdot \mathfrak{h}_i(\mathfrak{D}_E)(j)$. By restriction to $\mathfrak{h}_1(\mathfrak{D}_E)(j)$ we obtain a $\text{Gal}(E/E'_1)$ -invariant complement to $[Y_1, \mathfrak{h}_1(\mathfrak{D}_E)(j-2)]$ in $\mathfrak{h}_1(\mathfrak{D}_E)(j)$. This yields a contradiction by the above-treated absolutely almost simple cases.

Proof of (III) being equivalent to (IV), if (I) and (II) hold: First note that given a module $L_{\mathfrak{D}_E}$ satisfying Property (III) of Lemma 24, we may look at the image of either side of the equality of (III) inside $\mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+} = \mathfrak{g}(\kappa_E)$ to get

$$\dim_{\kappa_E} \mathfrak{g}(\kappa_E) - \dim_{\kappa_E} [Y, \mathfrak{g}(\kappa_E)] = \text{rk}_{\mathfrak{D}_E} L_{\mathfrak{D}_E} = \text{rk } G,$$

so that p is \mathfrak{g} -good. Thus we now assume that p is \mathfrak{g} -good. As the image $\overline{L}_{\mathfrak{D}_E}$ of $L_{\mathfrak{D}_E}$ in $\mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+} \simeq \mathfrak{g}(\kappa_E)$ is a complement in $\mathfrak{b}^-(\kappa_E)$ for $[\overline{Y}, \mathfrak{n}^-(\kappa_E)]$, we deduce (by Definition 15 and dimension counting) that $\overline{L}_{\mathfrak{D}_E}$ is a complement in $\mathfrak{g}(\kappa_E)$ for $[\overline{Y}, \mathfrak{g}(\kappa_E)]$, and hence Property (III) follows. \square

COROLLARY 26. *Let G be not too wild. If p is \mathfrak{g} -good and $\overline{\text{ad } Y} : \mathfrak{g}_{x,r}(0)/\mathfrak{g}_{x,r+}(0) \rightarrow \mathfrak{g}_{x,r}(2)/\mathfrak{g}_{x,r+}(2)$ is surjective for all $r \in \mathbb{R}$, then p is \mathfrak{g} - F -good.*

Moreover, if G is tamely ramified, then p is \mathfrak{g} - F -good if and only if p is \mathfrak{g} -good.

Proof. If E/F is tame, then the claim follows from Lemma 22 (whose isomorphism preserves the $\text{Ad} \circ \lambda$ -eigenspaces) and the fact that the map $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \rightarrow (\mathfrak{g}(K)_{x,r}/\mathfrak{g}(K)_{x,r+})^{\text{Gal}(K/F)}$ is an isomorphism.

Suppose now that p is \mathfrak{g} -good. Then we have for $j < 0$ that $\overline{\text{ad } Y}$ sends $\mathfrak{g}(E)_{x,0}(j)/\mathfrak{g}(E)_{x,0+}(j)$ injectively to $\mathfrak{g}(E)_{x,0}(j+2)/\mathfrak{g}(E)_{x,0+}(j+2)$. Using the injection ξ_i from Lemma 22, we can embed $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \text{ad } \overline{Y}$ -equivariantly into $\mathfrak{g}(E)_{x,0}/\mathfrak{g}(E)_{x,0+}$, and this embedding preserves the decomposition into $\text{Ad} \circ \lambda$ -eigenspaces. Hence the action $\text{ad } \overline{Y}$ is injective on $\mathfrak{g}_{x,r}(j)/\mathfrak{g}_{x,r+}(j)$ for $j < 0$. Next we need to show that $\overline{\text{ad } Y}$ maps $\mathfrak{g}_{x,r}(j)/\mathfrak{g}_{x,r+}(j)$ surjectively onto $\mathfrak{g}_{x,r}(j+2)/\mathfrak{g}_{x,r+}(j+2)$ for $j > 0$. To see this, we use the notation from Step 2 of the proof of Lemma 24, and let $\sum_{\sigma \in \mathcal{Y}_i} \sigma \cdot X$ be an element of $\mathfrak{g}_{x,r}(j+2)$ (i fixed), where $X \in \mathfrak{h}_i(E'_i)_{x_i, re'_i}(j+2) \subset \mathfrak{g}(E)_{x, re}(j+2)$, where e'_i is the ramification degree of E'_i/F , and x_i is the projection of x into the building of $H_i(E'_i)$. By the equivalence of \mathfrak{g} -good and \mathfrak{g} - F -good for the tamely ramified group H_i , there exists X' in $\mathfrak{h}_i(E'_i)_{x_i, re'_i}(j)$ such that $X + \mathfrak{h}_i(E'_i)_{x_i, re'_i}(j+2) = [Y_i, X'] +$

$\mathfrak{h}_i(E'_i)_{x_i, re'_i + (j+2)}$, and hence $\sum_{\mathcal{Y}_i} \sigma \cdot X + \mathfrak{g}_{x,r+}(j+2) = [Y, \sum_{\mathcal{Y}_i} \sigma \cdot X'] + \mathfrak{g}_{x,r+}(j+2)$. As $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ is spanned by elements of the form $\sum_{\mathcal{Y}_i} \sigma \cdot X + \mathfrak{g}_{x,r+}$, we deduce the claim. Thus, if $\text{ad } Y : \mathfrak{g}_{x,r}(0)/\mathfrak{g}_{x,r+}(0) \rightarrow \mathfrak{g}_{x,r}(2)/\mathfrak{g}_{x,r+}(2)$ is surjective, we obtain that p is \mathfrak{g} - F -good, as desired. \square

REMARK 27. In Corollary 26, the condition that $\overline{\text{ad } Y} : \mathfrak{g}_{x,r}(0)/\mathfrak{g}_{x,r+}(0) \rightarrow \mathfrak{g}_{x,r}(2)/\mathfrak{g}_{x,r+}(2)$ be surjective is automatic if G is semisimple, as we will explain below. In fact, we do not need to assume that G is semisimple. Instead, consider the identity component Z^0 of the center of G , which is canonically an \mathfrak{O}_E -torus and hence gives its Lie algebra \mathfrak{z} an \mathfrak{O}_E -scheme structure $\mathfrak{z}_{\mathfrak{O}_E}$. Let us assume that $\mathfrak{z}_{\mathfrak{O}_E}(\mathfrak{O}_E)$ has an $\mathfrak{O}_E[\text{Gal}(E/F)]$ -invariant complement \mathfrak{s} in $\mathfrak{t}(\mathfrak{O}_E)$. Of course, this is trivially satisfied if G is semisimple.

If we can show that the morphism $\text{ad } Y : \mathfrak{s} \rightarrow \mathfrak{g}(2)(\mathfrak{O}_E)$ of $\mathfrak{O}_E[\text{Gal}(E/F)]$ -modules is an isomorphism, then, multiplying by ϖ_E^j for any given j and taking $\text{Gal}(E/F)$ -invariants, we would get from Remark 19(i) that $\text{ad } Y(\mathfrak{g}_{x,j/e}(0)) \supset \text{ad } Y((\varpi_E^j \mathfrak{s})^{\text{Gal}(E/F)}) = \mathfrak{g}_{x,j/e}(2)$. Since j is arbitrary, and since $\mathfrak{g}_{x,r} = \mathfrak{g}_{x,[re]/e}$ for all r (use Remark 19(i) and that $\mathfrak{g}(E)_{x,re} = \mathfrak{g}(E)_{x,[re]}$ because x is hyperspecial over E), it would then follow that for all r , $\overline{\text{ad } Y} : \mathfrak{g}_{x,r}(0)/\mathfrak{g}_{x,r+}(0) \rightarrow \mathfrak{g}_{x,r}(2)/\mathfrak{g}_{x,r+}(2)$ is surjective. Thus, it suffices to show that the morphism $\text{ad } Y : \mathfrak{s} \rightarrow \mathfrak{g}(2)(\mathfrak{O}_E)$ of \mathfrak{O}_E -modules is an isomorphism, or equivalently, surjective (since $\text{rk}_{\mathfrak{O}_E} \mathfrak{s} = \text{rk}_{\mathfrak{O}_E} \mathfrak{g}(2)(\mathfrak{O}_E)$). But this follows since the map $\text{ad } Y : \mathfrak{g}(0)(\mathfrak{O}_E) \rightarrow \mathfrak{g}(2)(\mathfrak{O}_E)$ is surjective (p being \mathfrak{g} -good), with kernel containing $\mathfrak{z}_{\mathfrak{O}_E}(\mathfrak{O}_E)$.

Thus, if G is not too wild and semisimple (or, more generally, \mathfrak{z} has a suitable complement in \mathfrak{t}), and p is \mathfrak{g} -good, then p is \mathfrak{g} - F -good.

REMARK 28. We remark that p can be \mathfrak{g} - F -good without being \mathfrak{g} -good. This occurs, for example, if $G = \text{Res}_{E/F}(\text{SL}_2 \times \mathbb{G}_m)/\mu_2$ with $E = \mathbb{Q}_2[\sqrt{2}]$, $F = \mathbb{Q}_2$, and where the embedding $\mu_2 \hookrightarrow \text{Res}_{E/F}(\text{SL}_2 \times \mathbb{G}_m)$ is obtained by composing the diagonal embedding $\mu_2 \hookrightarrow \text{SL}_2 \times \mathbb{G}_m$ with the ‘diagonal’ inclusion $\text{SL}_2 \times \mathbb{G}_m \hookrightarrow \text{Res}_{E/F}(\text{SL}_2 \times \mathbb{G}_m)$ (which at the level of R -points, for an F -algebra R , corresponds to the inclusion $(\text{SL}_2 \times \mathbb{G}_m)(R) \hookrightarrow (\text{SL}_2 \times \mathbb{G}_m)(E \otimes_F R)$ induced by the R -algebra embedding $R \hookrightarrow E \otimes_F R, r \mapsto 1 \otimes r$).

COROLLARY 29. Suppose that G is not too wild and p is \mathfrak{n}^- -good. There exists a subspace $L_F \subset \mathfrak{b}^-(F)$ such that

- (i) L_F is a λ -invariant complement in $\mathfrak{b}^-(F)$ for $[Y, \mathfrak{n}^-(F)]$. Hence $Y + L_F$ is a section for $\mathfrak{g} \rightarrow \mathfrak{g}/G$ over F .
- (ii) The set of topologically nilpotent elements in $Y + L_F$ equals $Y + L_{0+}$, with $L_{0+} := L_F \cap \mathfrak{g}_{x,0+}$.

If we assume that p is \mathfrak{g} - F -good, then we can choose L_F such that

- (iii) $\mathfrak{g}_{x,r} = [Y, \mathfrak{g}_{x,r}] + L_r + \mathfrak{g}_{x,r+}$ for all $r \in \mathbb{R}$, where $L_r := L_F \cap \mathfrak{g}_{x,r}$.

Proof. Assuming G is not too wild and p is \mathfrak{n}^- -good puts at our disposal a lattice $L_{\mathfrak{O}_E}$ satisfying (I) of Lemma 24. The E -span of $L_{\mathfrak{O}_E}$ gets an F -structure L_F by Galois descent (see [33, Proposition 11.1.4]), and Condition (i) is easy to check. Since $L_{\mathfrak{O}_E}$ satisfies condition (I) of Lemma 24, Proposition 10 gives that $Y + L_{\mathfrak{O}_E} \rightarrow \mathfrak{g}/G$ is an isomorphism of schemes over \mathfrak{O}_E (where we use the \mathfrak{O}_E -structure arising from our fixed pinning). Therefore, Condition (ii) then follows from Lemma 5 and Remark 19(i), which applies as G is not too wild.

Suppose now that p is \mathfrak{g} - F -good. We work with the L_F constructed above. We have $\mathfrak{g}_{x,r} \supset [Y, \mathfrak{g}_{x,r}] + L_r + \mathfrak{g}_{x,r+}$, and hence $\dim_{\kappa} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \geq \dim_{\kappa} [Y, \mathfrak{g}_{x,r}]/\mathfrak{g}_{x,r+} + \dim_{\kappa} (L_r + \mathfrak{g}_{x,r+})/\mathfrak{g}_{x,r+}$ for all r , because $[Y, \mathfrak{g}_{x,r}]/\mathfrak{g}_{x,r+}$ and $(L_r + \mathfrak{g}_{x,r+})/\mathfrak{g}_{x,r+}$ have trivial intersection

by our construction of L_F . Since p is \mathfrak{g} - F -good we have (see Remark 21):

$$\begin{aligned} \sum_{i=1}^m (\dim_{\kappa} \mathfrak{g}_{x,r_i}/\mathfrak{g}_{x,r_i+} - \dim_{\kappa} [Y, \mathfrak{g}_{x,r_i}/\mathfrak{g}_{x,r_i+}]) &= \sum_{i=1}^m \dim_{\kappa} \mathfrak{g}_{x,r_i}(0)/\mathfrak{g}_{x,r_i+}(0) = \dim \mathfrak{t} = \dim_F L_F \\ &= \sum_{i=1}^m \dim_{\kappa} (L_{r_i} + \mathfrak{g}_{x,r_i+})/\mathfrak{g}_{x,r_i+}. \end{aligned}$$

Hence $\dim_{\kappa} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} = \dim_{\kappa} [Y, \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}] + \dim_{\kappa} (L_r + \mathfrak{g}_{x,r+})/\mathfrak{g}_{x,r+}$ and therefore $\mathfrak{g}_{x,r} = [Y, \mathfrak{g}_{x,r}] + L_r + \mathfrak{g}_{x,r+}$ for all $r \in \mathbb{R}$. \square

NOTATION 30. Given L_F as in Corollary 29, and any $r \in \mathbb{R}$, we will write L_r for $L_F \cap \mathfrak{g}_{x,r}$ and L_{r+} for $L_F \cap \mathfrak{g}_{x,r+}$, continuing with (and slightly extending) the notation introduced in (ii) and (iii) of the corollary.

2.8. Relating $Y + \mathfrak{g}_{x,r}$ with a Kostant section

If $K = E$, so that G is unramified, then x is a hyperspecial point of G , realizing G as a reductive group over \mathfrak{O} . In such a situation, we supply \mathfrak{q} with the \mathfrak{O} -structure determined by the containment $\mathfrak{O}[\mathfrak{g}]^G \subset F[\mathfrak{g}]^G$, and write $\mathfrak{q}(m)$ for the preimage of $\chi(0) \in \mathfrak{q}(\mathfrak{O}/\varpi^m \mathfrak{O})$ under the obvious map $\mathfrak{q}(\mathfrak{O}) \rightarrow \mathfrak{q}(\mathfrak{O}/\varpi^m \mathfrak{O})$ (see Notation 4).

DEFINITION 31. Say that G satisfies *condition (T)* if there is a tame extension E'/F such that $G_{E'}$ contains a maximal torus that is induced.

We've adapted terminology from [37].

LEMMA 32. Assume that G is not too wild and satisfies condition (T).

- (i) Suppose p is \mathfrak{g} - F -good, so that we may and do choose L_F as in Corollary 29. For $r > 0$, $Y + \mathfrak{g}_{x,r} = {}^{G_{x,r}}(Y + L_r)$ (where, for $J \subset G(F)$ and $\Omega \subset \mathfrak{g}(F)$, we write ${}^J \Omega$ for $\text{Ad}(J)(\Omega)$);
- (ii) In the situation of (i), suppose $K = E$, so that G is unramified. Then for $m \in \mathbb{N}$ we have

$$\chi(Y + \mathfrak{g}(F)_{x,m}) = \mathfrak{q}(m).$$

Proof. Part (ii) follows from (i) together with Proposition 10.

Let us prove (i) exactly as in [11, Lemma 5.2.1] (whose setting is far less restrictive, but requires p to be zero or large), which is in turn inspired by [35, §IX.4]. Only for the reader's convenience, we give the details.

Only the containment ' \subseteq ' is nontrivial. By condition (iii) in Corollary 29, we have that for each l , $\mathfrak{g}_{x,l} = [Y, \mathfrak{g}_{x,l}] + L_l + \mathfrak{g}_{x,l+}$. Hence it is enough to prove that for each $l \geq r$,

$$Y + L_r + [Y, \mathfrak{g}_{x,l}] \subset {}^{G_{x,l}}(Y + L_r + \mathfrak{g}_{x,l+}). \quad (2.10)$$

Let $Y + C + [Y, P]$ belong to the left-hand side, with $C \in L_r$ and $P \in \mathfrak{g}_{x,l}$. We wish to find $h \in G_{x,l}$ such that $\text{Ad } h(Y + C + [Y, P]) \in Y + L_r + \mathfrak{g}_{x,l+}$.

For this we use a mock exponential map. Let $\varphi_l : \mathfrak{g}_{x,l} \rightarrow G_{x,l}$ be as constructed in [1, §§ 1.3–1.5]. (Note that the assumption of G satisfying condition (T) is necessary to ensure that such a map exists, contrary to the claims in [1, 25].) We set $h = \varphi_l(-P)$. Since

$$\begin{aligned} \text{Ad } h(Y + C + [Y, P]) &= Y + C + (\text{Ad } h(Y) - Y + [Y, P]) + (\text{Ad } h(C) - C) + (\text{Ad } h([Y, P]) - [Y, P]), \end{aligned}$$

it suffices to show that each of the three parenthetical terms on the right-hand side of the above equation belongs to $\mathfrak{g}_{x,l+}$.

This follows from [1, Proposition 1.6.3], together with the fact that $[\mathfrak{g}_{x,a}, \mathfrak{g}_{x,b}] \subseteq \mathfrak{g}_{x,a+b}$ for all $a, b \in \mathbb{R}$. \square

2.9. The main result of [4] under our assumptions

The following result amounts to (a slight sharpening of) [4, Proposition 1] but with much milder hypotheses.

PROPOSITION 33. *Assume G is not too wild, and p is \mathfrak{n}^- -good.*

(i) *Let $Z \in \mathfrak{g}(F)$. Then Z is $G(\overline{F})$ -conjugate to an element of $Y + \mathfrak{g}_{x,0+}$ if and only if it is regular and topologically nilpotent.*

(ii) *If G satisfies condition (T), and p is \mathfrak{g} - F -good, then for any regular $Z \in \mathfrak{g}(F)_{\text{tn}}$, $\text{Ad } G(\overline{F})(Z) \cap (Y + \mathfrak{g}_{x,0+})$ is a single orbit under $G_{x,0+}$.*

Proof. Let $X \in Y + \mathfrak{g}_{x,0+} \subset \mathfrak{g}(\mathfrak{D}_E)$, and let $C_{G_{\mathfrak{D}_E}}(X)$ be the centralizer scheme of X in $G_{\mathfrak{D}_E}$. By [14, Exposé VI_B, Proposition 4.1], $\dim C_{G_E}(X) = \dim C_{G_{\mathfrak{D}_E}}(X)_E \leq \dim C_{G_{\mathfrak{D}_E}}(X)_{\kappa_E} = \dim C_{G_{\kappa_E}}(\overline{Y})$, where \overline{Y} is the image of Y in $\mathfrak{g}(\kappa_E)$. Hence, since \overline{Y} is regular [28, Lemma 3.1.1], we deduce that X is regular. Thus, if Z is conjugate to an element in $Y + \mathfrak{g}_{x,0+}$, then it is regular. Moreover, that every element of $Y + \mathfrak{g}_{x,0+}$ is topologically nilpotent follows from base-changing to a field over which G splits and applying Lemma 5 and Remark 19(i).

Suppose now that Z is regular. Following the reasoning of [28] (around equation (3.1.1)), we see that $Y + L_F$ consists entirely of regular elements. More precisely, it follows from the facts that Y is regular, that the locus of regular elements is open and that there is a contracting action of \mathbb{G}_m on $Y + (L_F \otimes_F E)$, which was described in the proof of Lemma 12. Thus, Z is $G(\overline{F})$ -conjugate to the unique (regular) element $Z' \in Y + L_F$ such that $\chi(Z) = \chi(Z')$. Therefore, $Z \in \mathfrak{g}(F)_{\text{tn}}$ if and only if $Z' \in \mathfrak{g}(F)_{\text{tn}}$, which by Condition (ii) in Corollary 29 is equivalent to $Z' \in Y + L_{0+} \subset Y + \mathfrak{g}_{x,0+}$. This gives (i).

Lemma 32 now gives (ii): since the proof of (i) shows that $Z \in \mathfrak{g}(F)_{\text{tn}}$ if and only if $\text{Ad } G(\overline{F})(Z)$ meets $Y + L_{0+}$, the assertion that $\text{Ad } G(\overline{F})(Z) \cap (Y + \mathfrak{g}_{x,0+})$ is a single $G_{x,0+}$ -orbit is equivalent to saying that $Y + \mathfrak{g}_{x,0+} = {}^{G_{x,0+}}(Y + L_{0+})$. \square

REMARK 34. Although we do not need it, we mention in passing that Proposition 33 holds for slightly more general groups. First, it holds when G is an arbitrary torus (that is, not necessarily satisfying condition (T)), because of our definition of ‘topologically nilpotent’. Second, if the proposition holds for two groups, then it holds for their direct product. Third, if the proposition holds for a group, then it holds for the image of the group under any isogeny whose schematic kernel has order not divisible by p .

REMARK 35. Assume that p is \mathfrak{g} -good and G is not too wild. Let $\mathfrak{g}(F)'_{\text{tn}}$ denote the union of the lattices $\mathfrak{g}_{z,0+}$ as z varies over $\mathcal{B}(G)$. Recall that in [4], unlike in Definition 1, an element of $\mathfrak{g}(F)$ is called ‘topologically nilpotent’ precisely when it belongs to $\mathfrak{g}(F)'_{\text{tn}}$. We now show that under our hypotheses, $\mathfrak{g}(F)_{\text{tn}} = \mathfrak{g}(F)'_{\text{tn}}$. The set $\mathfrak{g}(F)_{\text{tn}}$ is open and closed in $\mathfrak{g}(F)$, as can be seen for instance by base-changing to E and using Lemma 5. To see that $\mathfrak{g}(F)'_{\text{tn}}$ is also open and closed in $\mathfrak{g}(F)$, note that in the definition of this set, we may restrict our union to barycenters z of alcoves in $\mathcal{B}(G)$, which shows that for some positive $\varepsilon > 0$, $\mathfrak{g}(F)'_{\text{tn}} = \mathfrak{g}_\varepsilon$ in the notation of [2, Corollary 3.4.3], which we can then apply. Thus, it suffices to show that the set of regular semisimple elements in $\mathfrak{g}(F)_{\text{tn}}$ is the same as the set of regular

semisimple elements in $\mathfrak{g}(F)'_{\text{tn}}$ (here we are using that the regular semisimple elements are dense in $\mathfrak{g}(F)$; this is an easy consequence of the fact that $d\alpha$ does not vanish for any root α , which follows, as mentioned in [28, Remark 2.2.1(1)], from p being \mathfrak{g} -good). If $Z \in \mathfrak{g}(F)_{\text{tn}}$ is regular semisimple, then by Proposition 33, Z has a $G(\overline{F})$ -conjugate in $Y + \mathfrak{g}_{x,0+}$, which by [2, Corollary 3.2.6] is contained in $\mathfrak{g}(F)'_{\text{tn}}$. Hence $Z \in \mathfrak{g}(F)'_{\text{tn}}$ as well, since $\mathfrak{g}(F)'_{\text{tn}}$ is a union of stable orbits. (This follows from [6, Lemma 8.5]. Alternatively, use an argument combining [4, Hypothesis 3] and [3, Lemma 2.2.5], as in the proof of the ‘ \Rightarrow ’ implication of [4, Proposition 1]). Conversely, suppose $Z \in \mathfrak{g}(F)'_{\text{tn}}$ is regular semisimple. Then the identity component T_Z of the centralizer of Z in G is a maximal torus. For an extension E_1 of F splitting T_Z , we have $Z \in \mathfrak{g}(E_1)'_{\text{tn}}$, so that $Z \in \mathfrak{t}_1(E_1)'_{\text{tn}}$ by [2, Theorem 3.1.2(2)], forcing $Z \in \mathfrak{g}(E_1)_{\text{tn}} \cap \mathfrak{g}(F) = \mathfrak{g}(F)_{\text{tn}}$.

3. Some pairs of matching functions

We now assume the following:

- HYPOTHESIS 36. (a) F is a finite extension of \mathbb{Q}_p .
 (b) p is \mathfrak{g} -good for G .
 (c) $p \neq 2$.
 (d) G satisfies condition (T) and is not too wild. Moreover, if $p = 3$, then, writing G_{sc} as a product of groups $\text{Res}_{E_i/F} H_i$, with H_i an absolutely almost simple group over E_i , each group H_i splits over a quadratic extension of E_i .
 (e) p does not divide the cardinality of the center of G_{sc} .

Henceforth, the absolute value $|\cdot|$ on F will be normalized, and $|\cdot|$ will continue to denote its unique extension to \overline{F} . For any finite extension E' of F in \overline{F} , let $|\cdot|_{E'}$ denote the extension to \overline{F} of the normalized absolute value on E' .

REMARK 37. It follows from Hypothesis 36(e) that for any Galois extension E/F over which G splits, $\text{Lie}(Z^0)(\mathfrak{O}_E)$ has a $\text{Gal}(E/F)$ -invariant complement over \mathfrak{O}_E . Therefore, by Remark 27, p is \mathfrak{g} - F -good.

REMARK 38. (a) We assume part (a) of Hypothesis 36 not only to make it easier to handle orbits and measures on orbits, but also because we have not yet been able to locate references stating a suitable level of generality in which [22, Theorem 5.1], which we will need later, may be applied.

(b) Parts (c) and (e) are superfluous unless G has a factor of type A_n .

(c) Part (d) only excludes a few cases beyond those already excluded by other parts of Hypothesis 36: We require $p \neq 3$ if some H_i is an unramified triality form of D_4 . (This makes p , if positive, a ‘very good prime’ in the sense of [6, §8.10].) We require this hypothesis, along with parts (c) and (e), in order to apply a Kazhdan–Varshavsky quasi-logarithm.

Let H be a (necessarily quasi-split) group underlying a fixed endoscopic datum for G , and assume that H satisfies Hypothesis 36 as well. Henceforth, we fix a Galois extension E/F that splits both G and H .

3.1. Comments on measures

We will have to work with Haar measures specified using differential forms, and follow the second paragraph of [23, §1.4]. For an algebraic group G_1 over F , recall that $G_{1,\overline{F}}$ denotes

its base change to \overline{F} . First recall [23, § 1.4] that, given an algebraic group G_1 over F and a highest-degree invariant differential form ω_1 on $G_{1,\overline{F}}$, we can attach a Haar measure $|\omega_1|$ on $G_1(F)$ by choosing any $\mu_1 \in \overline{F}^\times$ such that $\mu_1\omega_1$ is defined over F (such a μ_1 exists by Hilbert's Theorem 90), and setting $|\omega_1| := |\mu_1|^{-1}|\mu_1\omega_1|$. One similarly obtains a Haar measure on $G_1(E')$ for every finite extension E'/F (using ω_1 and the normalized absolute value $|\cdot|_{E'}$ on E'), and this measure will be denoted by $|\omega_1|_{E'}$. Choose highest-degree differential forms ω_G, ω_H and ω_T on G, H and T respectively, and use these to fix Haar measures dg, dh and dt on $G(F), H(F)$ and $T(F)$, respectively, in the manner just described. By transport of structure from T via inner automorphisms, we can choose highest-degree forms $\omega_{T'}$ on each maximal torus T' of $G_{\overline{F}}$ (well defined up to scaling by $\mathfrak{O}_{\overline{F}}^\times$). Further, the endoscopic datum also allows us to transfer ω_T to a highest-degree differential form $\omega_{T''}$ on each maximal torus T'' of $H_{\overline{F}}$ (see [23, § 1.4]). For a maximal torus T' of G or H defined over F , we therefore get an associated measure $dt' = |\omega_{T'}|$ on $T'(F)$, which (unlike $\omega_{T'}$) does not depend on any choice other than, of course, that of ω_T .

REMARK 39. From now on, until § 3.4, we will state and prove certain results for G . Since H satisfies the same hypotheses as G , we may and will later apply them in the context of H , too.

Recall that for each $y \in \mathcal{B}(G)$ and $r \in \mathbb{R}$, we also have a Moy–Prasad lattice $\mathfrak{g}_{y,r}^* \subset \mathfrak{g}^*(F)$ given by:

$$\mathfrak{g}_{y,r}^* = \{ \Upsilon \in \mathfrak{g}^*(F) \mid \Upsilon(\mathfrak{g}_{y,(-r)+}) \subseteq \varpi \mathfrak{O}_F \}. \quad (3.1)$$

By [5, Proposition 4.1], thanks to p being \mathfrak{g} -good, there exists an $\text{Ad } G$ -invariant symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} that induces an identification of $\mathfrak{g}_{y,r}$ with $\mathfrak{g}_{y,r}^*$ for each $y \in \mathcal{B}(G)$ and $r \in \mathbb{R}$. Fix one such.

REMARK 40. In [5] such a form $\langle \cdot, \cdot \rangle$ is constructed as the restriction to $\mathfrak{g}(F)$ of a bilinear form on $\mathfrak{g}(E_1)$ satisfying analogous properties over E_1 , where E_1/F is an extension that splits G . Thus, we may and do assume that for each $y \in \mathcal{B}(G, E)$, $\langle \cdot, \cdot \rangle$ induces an identification of the lattice $\mathfrak{g}(E)_{y,r} \subset \mathfrak{g}(E)$ with the lattice $\mathfrak{g}^*(E)_{y,r} \subset \mathfrak{g}^*(E)$. In particular, if y is hyperspecial over E , $\langle \cdot, \cdot \rangle$ induces a perfect pairing on $\mathfrak{g}(E)_{y,0}$.

Now let $X \in \mathfrak{g}(\overline{F})$ be regular semisimple, so that its centralizer $T_X \subset G_{\overline{F}}$ is a maximal torus. The Lie algebra \mathfrak{t}_X of T_X is also the kernel of $\text{ad } X$. We now have two $G_{\overline{F}}$ -invariant top-degree differential forms (which are well-defined modulo $\mathfrak{O}_{\overline{F}}^\times$ -scaling) on the $G_{\overline{F}}$ -orbit of X , which may and will be identified with $G_{\overline{F}}/T_X$. The first is $\omega_X := \omega_G/\omega_{T_X}$, and the second is the differential form ω'_X arising from the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_X$ on the tangent space $\mathfrak{g}_{\overline{F}}/\mathfrak{t}_X$ to the variety $G_{\overline{F}}/T_X$ at $1 \cdot T_X$ induced by the degenerate symplectic pairing on $\mathfrak{g}_{\overline{F}}$ defined by $\langle v, w \rangle_X := \langle X, [v, w] \rangle = \langle [X, v], w \rangle$. Thus, if e_1, \dots, e_{2r} is an ordered symplectic basis for $\langle \cdot, \cdot \rangle_X$, that is, $\langle e_i, e_j \rangle_X = \delta_{i(2r+1-j)}$ for $1 \leq i \leq r$, then ω'_X takes the value one on $e_1 \wedge \dots \wedge e_{2r}$. If X and each e_i are defined over F and the \mathfrak{O} -lattice spanned by the e_i in $\mathfrak{g}(F)/\mathfrak{t}_X(F)$ is the image of an \mathfrak{O} -lattice L in $\mathfrak{g}(F)$, then the measure $|\omega'_X|$ on $G(F)/T_X(F)$ is the quotient of the measures on $G(F)$ and $T_X(F)$ corresponding to the measures on $\mathfrak{g}(F)$ and $\mathfrak{t}_X(F)$ normalized by L and $L \cap \mathfrak{t}_X(F)$, respectively.

REMARK 41. For any finite extension E'/F in \overline{F} , given any differential form ω on $\mathfrak{g} \times_F \overline{F}$, the measure $|\omega|_{E'}$ on $\mathfrak{g}(E')$ can be described as follows. Given a lattice $L \subset \mathfrak{g}(E')$, let a denote a generator of the lattice determined by L inside the top exterior power $\bigwedge_{E'}^{\dim \mathfrak{g}} \mathfrak{g}(E')$. For example,

a could be $e_1 \wedge \cdots \wedge e_{\dim \mathfrak{g}}$ for any $\mathfrak{D}_{E'}$ -basis $e_1, \dots, e_{\dim \mathfrak{g}}$ of L . Let $\tilde{m}(L, \omega) \in \overline{F}^\times / \mathfrak{D}_{\overline{F}}^\times$ denote the image of $\omega(a)$. This is independent of our choice of a . Then

$$\text{meas}(L, |\omega|_{E'}) = |\tilde{m}(L, \omega)|_{E'} = |\tilde{m}(L, \omega)|_F^{f_{E'/F}},$$

where $f_{E'/F}$ denotes the residue degree of E'/F . We define

$$m(L, |\omega|) := \text{meas}(L, |\omega|_{E'})^{1/f_{E'/F}}.$$

Note that this definition has the following invariance property: if E', L, ω are as above, and $E'' \supset E'$ is a finite extension inside \overline{F} , then $m(L, |\omega|) = m(L \otimes_{\mathfrak{D}_{E'}} \mathfrak{D}_{E''}, |\omega|)$.

3.2. The relation between $|\omega_X|$ and $|\omega'_X|$

LEMMA 42. *The measures $|\omega_X|$ and $|\omega'_X|$ are related by*

$$|\omega'_X| = D_{\mathfrak{g}}(X)^{1/2} \frac{m(\mathfrak{t}(\mathfrak{D}_E), |\omega_T|)}{m(\mathfrak{g}(E)_{x,0}, |\omega_G|)},$$

where $D_{\mathfrak{g}}(X) = |\det(\text{ad } X; \mathfrak{g}/\mathfrak{t}_X)|$.

Proof. Using Remark 41 and that $|a\omega| = |a||\omega|$ for any top-degree form ω on $G_{\overline{F}}$ and $a \in \overline{F}^\times$, one sees that it suffices to prove:

$$\omega'_X \in \mathfrak{D}_{\overline{F}}^\times \cdot \det(\text{ad } X; \mathfrak{g}/\mathfrak{t}_X)^{1/2} \frac{\tilde{m}(\mathfrak{t}(\mathfrak{D}_E), \omega_T)}{\tilde{m}(\mathfrak{g}(E)_{x,0}, \omega_G)} \cdot \omega_X$$

(the choice of square root clearly does not matter). This formulation allows us to change the base field: since the assignments $X \mapsto \omega_X$ and $X \mapsto \omega'_X$ behave well with respect to $G(\overline{F})$ -conjugation (in the sense that $\omega_X \in \omega_{\text{Ad } g(X)} \cdot \mathfrak{D}_{\overline{F}}^\times$ and similarly for ω'_X), we may and do assume that $X \in \mathfrak{t}(E')$ for some finite extension E' of E contained in \overline{F} . Thus, $T_{E'} = T_{X, E'}$. For each root $\alpha \in R(B, T)$, by the construction of the form $\langle \cdot, \cdot \rangle$, there exists $a_\alpha \in \mathfrak{D}_{\overline{F}}^\times$ such that $\langle a_\alpha X_\alpha, X_{-\alpha} \rangle = 1$, where for this proof, we write $X_{-\alpha}$ to mean a fixed basis element for the root space $\mathfrak{g}_{-\alpha}(\mathfrak{D}_E)$ (recall that our fixed pinning realizes G as a split reductive group over \mathfrak{D}_E). Then an ordered symplectic basis $e_1, \dots, e_{\dim G - \text{rk } G}$ for $\langle \cdot, \cdot \rangle_X$ over E' may be chosen to have as its underlying set:

$$\{a_\alpha \cdot d\alpha(X)^{-1} X_\alpha \mid \alpha > 0\} \cup \{X_{-\alpha} \mid \alpha > 0\},$$

with $d\alpha$ denoting the derivative of α . The form ω'_X then takes the value 1 at $e_1 \wedge \cdots \wedge e_{\dim G - \text{rk } G}$, whereas, the image of $\omega_X(e_1 \wedge \cdots \wedge e_{\dim G - \text{rk } G})$ in $\overline{F}^\times / \mathfrak{D}_{\overline{F}}^\times$ equals

$$\det(\text{ad } X; \mathfrak{g}/\mathfrak{t}_X)^{-1/2} \frac{\tilde{m}(\mathfrak{g}(E)_{x,0} \otimes_{\mathfrak{D}_E} \mathfrak{D}_{E'}, \omega_G)}{\tilde{m}(\mathfrak{t}(\mathfrak{D}_{E'}), \omega_T)} = \det(\text{ad } X; \mathfrak{g}/\mathfrak{t}_X)^{-1/2} \frac{\tilde{m}(\mathfrak{g}(E)_{x,0}, \omega_G)}{\tilde{m}(\mathfrak{t}(\mathfrak{D}_E), \omega_T)}$$

(use that $\prod_{\alpha > 0} d\alpha(X) \in \mathfrak{D}_{\overline{F}}^\times \cdot \det(\text{ad } X; \mathfrak{g}/\mathfrak{t}_X)^{1/2}$, and also use Remark 41 again), proving the lemma. \square

3.3. Studying ω'_X for semisimple $X \in Y + \mathfrak{g}_{x,0+}$

NOTATION 43. If L is a lattice (of full rank) in a finite-dimensional vector space V over F with a symmetric or alternating bilinear form B , then we set (suppressing dependence on B for lightness of notation)

$$L^\perp = \{v \in V \mid B(v, L) \subseteq \mathfrak{D}\}.$$

LEMMA 44. *Suppose $X \in Y + \mathfrak{g}_{x,0+}$ is semisimple. Then, in the symplectic space $(\mathfrak{g}(F)/\mathfrak{t}_X(F), \langle \cdot, \cdot \rangle_X)$,*

$$(\mathfrak{g}_{x,0+}/(\mathfrak{g}_{x,0+} \cap \mathfrak{t}_X(F)))^\perp = \mathfrak{g}_{x,-1}/(\mathfrak{g}_{x,-1} \cap \mathfrak{t}_X(F)). \quad (3.2)$$

Proof. By equation (3.1) and the choice of $\langle \cdot, \cdot \rangle$, we have $\mathfrak{g}_{x,0+}^\perp = \varpi^{-1}\mathfrak{g}_{x,0} = \mathfrak{g}_{x,-1}$, with \perp being taken with respect to the bilinear form $\langle \cdot, \cdot \rangle$. This together with the fact that $\mathfrak{g}_{x,0+}$ is invariant under $\text{ad } X$ gives the relation ‘ \supseteq ’.

For the reverse inclusion, it is enough to show that any element $Z \in \mathfrak{g}(F)$ such that $[X, Z] \in \mathfrak{g}_{x,-1}^\perp = \mathfrak{g}_{x,0+}$ (\perp taken with respect to $\langle \cdot, \cdot \rangle$) necessarily belongs to $\mathfrak{t}_X(F) + \mathfrak{g}_{x,0+} = \ker \text{ad } X + \mathfrak{g}_{x,0+}$. In other words, it is enough to show that

$$[X, \mathfrak{g}_{x,0+}] = [X, \mathfrak{g}(F)] \cap \mathfrak{g}_{x,0+}. \quad (3.3)$$

This is equivalent to showing that the rank of the endomorphism $\overline{\text{ad } X}$ of $\mathfrak{g}_{x,0+}/\varpi\mathfrak{g}_{x,0+}$ induced by $\text{ad } X$, which *a priori* is at most $\dim_F[X, \mathfrak{g}(F)] = \dim \mathbf{G} - \text{rk } \mathbf{G}$, is actually equal to $\dim \mathbf{G} - \text{rk } \mathbf{G}$. Now $\overline{\text{ad } X}$ preserves the Moy–Prasad filtration of $\mathfrak{g}_{x,0+}/\varpi\mathfrak{g}_{x,0+} = \mathfrak{g}_{x,0+}/\mathfrak{g}_{x,1+}$ induced by the lattices $\mathfrak{g}_{x,r+}$ (for $0 \leq r < 1$), and we can consider the associated graded map $\overline{\text{ad } X}_{\text{gr}}$. *A priori*, the rank of $\overline{\text{ad } X}_{\text{gr}}$ is at most that of $\overline{\text{ad } X}$, so it suffices to show that the rank of $\overline{\text{ad } X}_{\text{gr}}$ equals $\dim_F[X, \mathfrak{g}(F)] = \dim \mathbf{G} - \text{rk } \mathbf{G}$. The map $\overline{\text{ad } X}_{\text{gr}}$ is the same as the associated graded map of the analogously defined endomorphism $\overline{\text{ad } Y}$. Since p is \mathfrak{g} - F -good by Remark 37, Corollary 29(iii) applies to show, using notation from there, that the codimension of the image of $\overline{\text{ad } Y}$ on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ equals $\dim_\kappa L_r/L_{r+}$. Thus, the rank of $\overline{\text{ad } X}_{\text{gr}}$ equals $\dim \mathfrak{g} - \dim_\kappa(L_{0+}/\varpi L_{0+})$, which equals $\dim \mathfrak{g} - \text{rk } \mathfrak{g}$. \square

3.4. Map of adjoint quotients

Recall that we are assuming that \mathbf{H} splits over E . Just as we did with \mathbf{G} in § 2.5, we fix a $\text{Gal}(\overline{F}/F)$ -stable pinning for \mathbf{H} over E and get a point $x_{\mathbf{H}}$ in the reduced Bruhat–Tits building $\mathcal{B}(\mathbf{H})$ of \mathbf{H} as well as a regular nilpotent element $Y_{\mathbf{H}} \in \mathfrak{h}(F)$.

Let $\chi_{\mathbf{H}} : \mathfrak{h} \rightarrow \mathfrak{q}_{\mathbf{H}}$ denote the adjoint quotient of \mathfrak{h} (as in Notation 4). Following the notational setup of [19, § 1.1.6], the transfer of stable conjugacy classes from \mathfrak{h} to \mathfrak{g} is described by a finite morphism $\nu : \mathfrak{q}_{\mathbf{H}} \rightarrow \mathfrak{q}$. Namely, for each maximal torus $T_{\mathbf{H}}$ of \mathbf{H} , there exists a certain stable conjugacy class of embeddings $\iota : T_{\mathbf{H}} \hookrightarrow \mathbf{G}$ each of which defines an isomorphism of $T_{\mathbf{H}}$ onto some maximal torus T' of \mathbf{G} (any such ι is an admissible embedding as named in [23, § 1.3]), and all of which satisfy that $\chi \circ d\iota = \nu \circ \chi_{\mathbf{H}}$.

Recall that functions $\phi \in C_c^\infty(\mathfrak{g}(F))$ and $\phi_{\mathbf{H}} \in C_c^\infty(\mathfrak{h}(F))$ are said to have matching orbital integrals if and only if for all \mathbf{G} -regular semisimple $X_{\mathbf{H}} \in \mathfrak{h}(F)$ (that is, $\nu \circ \chi_{\mathbf{H}}(X_{\mathbf{H}}) = \chi(X')$ for some regular semisimple $X' \in \mathfrak{g}(\overline{F})$), we have an equality

$$\sum_{X'_{\mathbf{H}}} I(X'_{\mathbf{H}}, \phi_{\mathbf{H}}, |\omega_{\mathbf{H}}/\omega_{T_{X'_{\mathbf{H}}}}|) = \sum_X \Delta'_0(X_{\mathbf{H}}, X) I(X, \phi, |\omega_{\mathbf{G}}/\omega_{T_X}|), \quad (3.4)$$

where we have written T_Z for the centralizer of Z in the appropriate group, I stands for *normalized* orbital integral, Δ'_0 denotes the transfer factor that excludes the term Δ_{IV} (which is accounted for by the normalization of orbital integrals), $X'_{\mathbf{H}}$ runs over a set of representatives for the $\mathbf{H}(F)$ -conjugacy classes in the stable conjugacy class of $X_{\mathbf{H}}$ (this stable conjugacy class equals $\chi_{\mathbf{H}}|_{\mathbf{H}(F)}^{-1}(\chi_{\mathbf{H}}(X_{\mathbf{H}}))$), and X runs over a set of representatives for the $\mathbf{G}(F)$ -conjugacy classes in $\chi|_{\mathbf{G}(F)}^{-1} \circ \nu \circ \chi_{\mathbf{H}}(X_{\mathbf{H}})$. Here the transfer factors need to be normalized, and we do so following [22], namely, normalizing them according to the F -conjugacy class of the pinning $(B^-, T, \{X_{-\alpha}\})$, where the vectors $X_{-\alpha}$ are normalized to satisfy $[X_{\alpha}, X_{-\alpha}] = H_{\alpha} = d\alpha^\vee(1)$.

REMARK 45. Let us state the consequence of this normalization of transfer factors that concerns us. Kottwitz [22, p. 128] associates a regular nilpotent element to the pinning $(B^-, T, \{X_{-\alpha}\})$. For us, this element equals Y . Corollary 29 gave us a Kostant section $Y + L_F$. Then [22, Theorem 5.1] says that $\Delta'_0(X_H, X) = 1$ whenever $X_H \in \mathfrak{h}(F)$ and $X \in Y + L_F \subset \mathfrak{g}(F)$ are regular semisimple elements that match in the sense that $\nu \circ \chi_H(X_H) = \chi(X)$. By Lemma 32(i), the previous sentence holds true with $Y + L_F$ replaced by $Y + \mathfrak{g}_{x,0+}$.

3.5. Some consequences of a Kazhdan–Varshavsky quasi-logarithm

To work with orbital integrals, we will need to relate the measure of $T_X(F) \cap G_{x,r}$ to that of $t_X(F) \cap \mathfrak{g}_{x,r}$. For this purpose alone, we will use a Kazhdan–Varshavsky quasi-logarithm.

REMARK 46. Identify $\mathcal{B}(G)$ with the Bruhat–Tits building of G_{sc} as well. Then

- (i) Since G satisfies Hypothesis 36(e), it follows from [6, Lemma 8.12] that the obvious isogeny $Z^0 \times G_{\text{sc}} \rightarrow G$ induces an isomorphism

$$Z_{0+}^0 \times G_{\text{sc},x,0+} \xrightarrow{\sim} G_{x,0+},$$

of p -adic analytic groups, where Z_{0+}^0 stands for $Z_{z,0+}^0$, z denoting the unique point in the reduced Bruhat–Tits building of Z^0 .

- (ii) By [6, Lemmas C.3 and C.4], we have an analytic isomorphism $\mathcal{L} : G_{\text{sc},x,0+} \rightarrow \mathfrak{g}_{\text{sc},x,0+}$ that is equivariant under $G_{\text{sc},x,0}$ -conjugation, and such that for all $r > 0$, $\mathcal{L}(G_{\text{sc},x,r}) = \mathfrak{g}_{\text{sc},x,r}$.

The consequences of Remark 46 that we wish to use are collected in the following corollary.

COROLLARY 47. *Let $X \in \mathfrak{g}(F)$ be regular semisimple with centralizer $T_{\text{sc},X} \subset G_{\text{sc}}$.*

- (i) \mathcal{L} takes $T_{\text{sc},X}(F) \cap G_{\text{sc},x,r}$ homeomorphically onto $t_{\text{sc},X}(F) \cap \mathfrak{g}_{\text{sc},x,r}$, for all $r > 0$.
- (ii) Suppose that $\mathfrak{g}_{\text{sc}}(F)$ and $G_{\text{sc}}(F)$ are given compatible measures, and that so are $t_{\text{sc},X}(F)$ and $T_{\text{sc},X}(F)$. Then, for $r > 0$, $\text{meas}(\mathfrak{g}_{\text{sc},x,r}) = \text{meas}(G_{\text{sc},x,r})$, and $\text{meas}(t_{\text{sc},X}(F) \cap \mathfrak{g}_{\text{sc},x,r}) = \text{meas}(T_{\text{sc},X}(F) \cap G_{\text{sc},x,r})$.

Proof. It is enough to see statements (i) and (ii) above when G is simply connected. For all $r > 0$, by Remark 46(ii), we have homeomorphisms $G_{x,r} \rightarrow \mathfrak{g}_{x,r}$ and $G_{x,0+} \rightarrow \mathfrak{g}_{x,0+}$. Now (i) follows from the conjugation equivariance of \mathcal{L} : picking any $t \in T_X(F)$ that is strongly regular in G (for example, $\exp(aX)$ with $a \in F$, $|a|$ small enough), we see that any given $g \in G_{x,0+}$ belongs to the kernel T_X of $\text{Int } t$ if and only if $\mathcal{L}(g) \in \mathfrak{g}_{x,0+}$ belongs to the kernel t_X of $\text{Ad } t$.

To see that (ii) follows too, it is enough to show that the top exterior power of the derivative of \mathcal{L} (respectively, $\mathcal{L}|_{T_X(F)}$) at each $g \in G_{x,r}$ (respectively, at each $t \in T_X(F) \cap G_{x,r}$), which is an endomorphism of a one-dimensional F -vector space, is a unit in \mathfrak{O} , or equivalently, $\mathfrak{O}_{\overline{F}}$. Let $E_X \subset \overline{F}$ be a field extension over which T_X splits. Now G_{E_X} gets an integral model $G_{\mathfrak{O}_{E_X}}$ from x , while T_{X,E_X} has a canonical integral model since it is split. We claim that the base changes of \mathcal{L} and $\mathcal{L}|_{T_X}$ to E_X extend to \mathfrak{O}_{E_X} -morphisms $G_{\mathfrak{O}_{E_X}} \rightarrow \mathfrak{g}_{\mathfrak{O}_{E_X}}$ and $T_{X,\mathfrak{O}_{E_X}} \rightarrow t_{X,\mathfrak{O}_{E_X}}$. Indeed, the assertion involving $G_{\mathfrak{O}_{E_X}}$ follows from \mathcal{L} being ‘defined over \mathfrak{O} ’ in the sense of [6, Appendix C.2] (see the last sentence of [6, C.2(a); 18, Definition 1.88(b) and Notation 1.8.6(b)]). This together with the conjugation equivariance of \mathcal{L} implies, as in (i), that \mathcal{L} takes $T_{\mathfrak{O}_{E_X}}(\mathfrak{O}_{\overline{F}})$ to $t_{\mathfrak{O}_{E_X}}(\mathfrak{O}_{\overline{F}})$, from which the assertion involving $T_{X,\mathfrak{O}_{E_X}}$ follows by [9, Proposition 1.7.6].

This already implies that the top exterior power of the derivative of \mathcal{L} (respectively, $\mathcal{L}|_{T_X(F)}$) at each $g \in G_{x,r}$ (respectively, at each $t \in T_X(F) \cap G_{x,r}$) belongs to $\mathfrak{O}_{\overline{F}}$, and it suffices to show that the image of this element in $\overline{\kappa}$ equals 1. But this image may be computed by base-changing to κ_{E_X} . However, g (respectively, t) has the identity for its image in $G_{\mathfrak{O}_{E_X}}(\kappa_{E_X})$ (respectively,

$T_{\mathfrak{D}_{E_X}}(\kappa_{E_X}))$, so that the result follows from the derivative of \mathcal{L} at the identity element being identity, by the very definition of a quasi-logarithm (see [6, C.1(a)]). \square

3.6. An orbital integral computation

LEMMA 48. *Suppose $X \in \mathfrak{g}(F)$ is regular semisimple. Then*

$$I(X, \mathbb{1}_{Y+\mathfrak{g}_{x,0+}}, |\omega_X|) = \begin{cases} c_G & \text{if } X \in \text{Ad } G(F)(Y + \mathfrak{g}_{x,0+}), \text{ and} \\ 0 & \text{otherwise,} \end{cases} \quad (3.5)$$

where

$$c_G = m(\mathfrak{t}(\mathfrak{D}_E), |\omega_T|)^{-1} m(\mathfrak{g}(E)_{x,0}, |\omega_G|) q^{-(\dim G - \text{rk } G + m)/2},$$

where m is the rank of the endomorphism of the κ -vector space $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ induced by $\text{ad } Y$.

Proof. We may and do assume $X \in Y + \mathfrak{g}_{x,0+}$. By Lemma 42 the left-hand side is given by the product of $m(\mathfrak{t}(\mathfrak{D}_E), |\omega_T|)^{-1} m(\mathfrak{g}(E)_{x,0}, |\omega_G|)$ and the unnormalized orbital integral $O(X, \mathbb{1}_{Y+\mathfrak{g}_{x,0+}}, |\omega'_X|)$. Hence by Proposition 33, the left-hand side equals

$$m(\mathfrak{t}(\mathfrak{D}_E), |\omega_T|)^{-1} m(\mathfrak{g}(E)_{x,0}, |\omega_G|) \cdot \text{meas}(G_{x,0+}/(G_{x,0+} \cap T_X(F)), |\omega'_X|).$$

Note that thanks to Hypothesis 36(e), the projection X_{sc} of X to $\mathfrak{g}_{\text{sc}}(F) \subset \mathfrak{g}(F)$ belongs to $Y + \mathfrak{g}_{\text{sc},x,0+}$, and

$$\text{meas}(G_{x,0+}/(G_{x,0+} \cap T_X(F)), |\omega'_X|) = \text{meas}(\mathfrak{g}_{\text{sc},x,0+}/(\mathfrak{g}_{\text{sc},x,0+} \cap \mathfrak{t}_{X_{\text{sc}}}(F)), |\omega'_{X_{\text{sc}}}|)$$

$$= [\mathfrak{g}_{\text{sc},x,-1}/(\mathfrak{g}_{\text{sc},x,-1} \cap \mathfrak{t}_X(F)) : \mathfrak{g}_{\text{sc},x,0+}/(\mathfrak{g}_{\text{sc},x,0+} \cap \mathfrak{t}_X(F))]^{-1/2}$$

(use Remark 46(i), Corollary 47(ii), and Lemma 44). Since the index of $\mathfrak{g}_{\text{sc},x,0}/(\mathfrak{g}_{\text{sc},x,0} \cap \mathfrak{t}_X(F))$ in $\mathfrak{g}_{\text{sc},x,-1}/(\mathfrak{g}_{\text{sc},x,-1} \cap \mathfrak{t}_X(F))$ equals $q^{\dim G - \text{rk } G}$, it now suffices to show that

$$\dim_{\kappa} \mathfrak{g}_{x,0}/(\mathfrak{g}_{x,0} \cap \mathfrak{t}_X(F) + \mathfrak{g}_{x,0+}) = \dim_{\kappa} ([Y, \mathfrak{g}_{x,0}] + \mathfrak{g}_{x,0+})/\mathfrak{g}_{x,0+}. \quad (3.6)$$

Thanks to equation (3.3), the left-hand side of the above equation is the rank of the κ -linear endomorphism of $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ induced by $\text{ad } X$. Now the lemma follows from the fact that $\text{ad } X$ and $\text{ad } Y$ induce the same map on $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$, since $X \in Y + \mathfrak{g}_{x,0+}$. \square

REMARK 49. Note that if p is large enough to satisfy the hypotheses of [11, § 4.2] (or equivalently, those of [12, § 2.2]) one might also be able to prove Lemma 48 as follows. Under this assumption, the main results of [4; 12, Theorem 2.1.5] imply that every $X \in \mathfrak{g}(F)_{\text{tn}}$ lies in the range of validity of the Shalika germ expansion for the function $\mathbb{1}_{Y+\mathfrak{g}_{x,0+}}$ near $0 \in \mathfrak{g}(F)$. This in turn implies Lemma 48 up to some constant independent of X . It should be possible to calculate this constant using the main result of [31] and comparing two different measures on $\text{Ad } G(F) \cdot Y$: the one considered by [31] and the one considered just before [12, Lemma 3.4.4].

3.7. The main result

Recall we have assumed that G and H satisfy Hypothesis 36.

PROPOSITION 50. *Let $\phi_H := \mathbb{1}_{Y_H+\mathfrak{h}_{x_H,0+}}$ and $\phi := \mathbb{1}_{Y+\mathfrak{g}_{x,0+}}$. Let c_G be as in Lemma 48, and let c_H be the analogous constant for H . Then $c_H^{-1}\phi_H$ and $c_G^{-1}\phi$ have matching orbital integrals.*

Proof. Let $X_H \in \mathfrak{h}(F)$ be G -regular semisimple. We need to prove an equality analogous to that in equation (3.4), which we may write as

$$\sum_{X'_H} I(X'_H, c_H^{-1}\phi_H, |\omega_H/\omega_{T_{X'_H}}|) = \sum_X \Delta'_0(X_H, X) I(X, c_G^{-1}\phi, |\omega_G/\omega_{T_X}|). \quad (3.7)$$

Since G is quasi-split, it is known that the sum on the right-hand side of equation (3.7) is nonempty (for example, this is an easy consequence of either of [21, Corollary 2.2 and Theorem 4.1]). It is also easy to see (see, for example, [19, § 1.1.7]) that for each X contributing to the right-hand side of that equation, one can choose an isomorphism $\iota: T_{H, X_H} \rightarrow T_X$ of the centralizers of X_H and X with $d\iota(X_H) = X$. By Definition 1, we conclude

$$X_H \in \mathfrak{h}(F)_{\text{tn}} \text{ if and only if } X \in \mathfrak{g}(F)_{\text{tn}}. \quad (3.8)$$

If $X_H \notin \mathfrak{h}(F)_{\text{tn}}$, then by (3.8), $X \notin \mathfrak{g}(F)_{\text{tn}}$ for any X that contributes to the right-hand side of equation (3.7). Recalling that $\text{Ad } G(F)(Y + \mathfrak{g}_{x,0+}) \subset \mathfrak{g}(F)_{\text{tn}}$ by Proposition 33(i), and similarly for H , both sides of equation (3.7) vanish when $X_H \notin \mathfrak{h}(F)_{\text{tn}}$.

Thus, now assume $X_H \in \mathfrak{h}(F)_{\text{tn}}$, so that, by (3.8), any X that contributes to the right-hand side of equation (3.7) belongs to $\mathfrak{g}(F)_{\text{tn}}$. By Proposition 33(ii) applied to H , exactly one X'_H contributes to the left-hand side of equation (3.7), say X_H , which may be assumed to lie in $Y_H + \mathfrak{h}_{x_H,0+}$. By the same result but applied to G , exactly one X contributes to the right-hand side of equation (3.7), and such an X may be assumed to lie in $Y + \mathfrak{g}_{x,0+}$. Thus, we now need to show that

$$I\left(X_H, c_H^{-1}\phi_H, |\omega_H/\omega_{T_{X_H}}|\right) = \Delta'_0(X_H, X) I(X, c_G^{-1}\phi, |\omega/\omega_{T_X}|).$$

By Remark 45, $\Delta'_0(X_H, X) = 1$. Now the result follows from Lemma 48. \square

3.8. Some more pairs of functions with matching orbital integrals

We expect that one can use the above result to produce more pairs of functions with matching orbital integrals. Let us sketch how this may be done.

(a) A ‘scaling’ argument as in [16, Proposition 3.2.2] (similar to an argument in [30, § 9]) shows that for all $l \in \mathbb{Z}$, up to a scalar (depending on l) the characteristic functions of $\varpi^{-l}Y + \mathfrak{g}_{x,(-l)+}$ and $\varpi^{-l}Y_H + \mathfrak{h}_{x_H,(-l)+}$ have matching orbital integrals.

(b) Let $l \in \mathbb{N}$. Recall the bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . It is explained in [34, § VIII.6] how to obtain from $\langle \cdot, \cdot \rangle$ a symmetric nondegenerate $\text{Ad } H$ -invariant bilinear form $\langle \cdot, \cdot \rangle_H$ on \mathfrak{h} . Using $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_H$ and an additive character Λ of F with conductor $\varpi\mathfrak{D}_F$ to define Fourier transforms on $\mathfrak{g}(F)$ and $\mathfrak{h}(F)$, we get two functions $\varphi_l \in C_c^\infty(\mathfrak{g}(F))$ and $\varphi_{H,l} \in C_c^\infty(\mathfrak{h}(F))$ that have matching orbital integrals, up to an explicit scalar (use [34, Conjecture 1], which has since been proved thanks to Waldspurger and Ngô, see [19, Theorem 4.1.3]). Here φ_l is supported on $\mathfrak{g}_{x,l}$, on which it is a scalar multiple of $X \mapsto \Lambda(\langle \varpi^{-l}Y, X \rangle)$, and $\varphi_{H,l}$ has a similar description.

(c) If we have suitable quasi-logarithms on H and G that are compatible with each other, and behave well with respect to transfer factors, one should be able to pull the functions in (b) above to the group level. Thus, if suitable hypotheses on the quasi-logarithm maps are satisfied (basically, statements along the lines of [19, Propositions 5.1.3 and 5.2.5(b) and (c)]), we should get that for $l \in \mathbb{N}$, $\tilde{\varphi}_l \in C_c^\infty(G(F))$ and $\tilde{\varphi}_{H,l} \in C_c^\infty(H(F))$ have matching orbital integrals, up to a scalar. Here $\tilde{\varphi}_l$ is supported on $G_{x,l}$, on which it is the inflation of the character of $G_{x,l}/G_{x,l+}$ obtained by composing the isomorphism $G_{x,l}/G_{x,l+} \cong \mathfrak{g}_{x,l}/\mathfrak{g}_{x,l+}$ of groups with (a scalar multiple of) $X \mapsto \Lambda(\langle \varpi^{-l}Y, X \rangle)$.

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